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Construction of Lyapunov functions from first integrals for stochastic stability analysis

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Abstract

This paper explores the application of first integrals in constructing Lyapunov functions for stability analysis of dynamical systems in stochastic domains. A key advantage of using first integrals is their ability to embed system-specific structural and physical information, distinguishing the resulting Lyapunov functions from generic positive definite functions with no intrinsic connection to the system. However, since first integrals do not inherently satisfy Lyapunov conditions, additional constraints—often with direct physical interpretations—must be introduced to ensure positive definiteness and suitable monotonic behavior. The method is demonstrated on three mechanical systems subjected to parametric noise: a nonlinear aeroelastic single-degree-of-freedom oscillator, a spherical pendulum with two first integrals, and a gyroscope with three first integrals.

Keywords: stochastic stability, Lyapunov function, first integrals, cyclic coordinates

1. Introduction

The study of dynamic stability in systems subjected to random excitation naturally extends the classical analysis where excitation is purely deterministic. Practical experience has shown that random excitation can have both stabilizing and destabilizing effects on system responses. In engineering applications, for example, it is common to introduce artificial turbulence to suppress vibrations—particularly in fields such as aerospace and civil engineering. These solutions are often developed heuristically and may lack a rigorous theoretical foundation. Conversely, random disturbances can interact hazardously with deterministic processes, significantly reducing system stability, even under seemingly benign conditions (e.g., icing or corrosion on cables, uneven road surfaces, or minor aerodynamic irregularities).

The concept of dynamic stability under random excitation (DSR), whether the randomness is additive or multiplicative, can be regarded as a generalization of the deterministic stability problem. Multiple definitions of DSR exist, reflecting differences in factors such as the type of response parameter considered, the interpretation of stability over time, and the properties of the input stochastic processes, e.g., [7, 11].

Among the tools used to investigate stochastic stability, a prominent method is the extension of Lyapunov's second method to systems under random excitation, as formalized in classical monographs such as [7] and [11]. Despite its foundational role in deterministic systems, Lyapunov's second method has been applied less frequently to stochastic problems. It is often regarded as better suited for gaining structural insight into system behavior rather than pinpointing specific stability thresholds. For a brief overview of Lyapunov methods in the context

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of stochastic differential equations, see [23]. A more general perspective on its use for random dynamical systems is offered in the review part of [2].

Nevertheless, when both sufficient and necessary conditions can be established for the existence of a Lyapunov function that corresponds to the desired stability type, the method becomes highly powerful for both theoretical investigations and practical applications. The construction of a Lyapunov function, when feasible, provides a comprehensive perspective on the system's stability, considering both global and local aspects. This versatility is the main strength of the Lyapunov's second method, though it also introduces certain challenges.

In the stochastic setting, the Lyapunov function can be interpreted as a generalized upper bound for the the total energy of the system. Conceptually, it corresponds to a positive definite function on the phase space (excluding the origin), with a negative total derivative with respect to time within the same region, corresponding to an energy-dissipating system. The system's evolution, regardless of initial conditions, is governed by this dissipation, possibly counteracted by any internal or external energy inputs, leading eventually to the convergence of all state variables to zero. This physical analogy underlies many of the practical methods used to construct Lyapunov functions in specific stochastic systems.

Finding appropriate Lyapunov functions for stochastic systems remains a non-trivial task and continues to be an active area of research. Early foundational work on the stochastic generalization of the Lyapunov method began with [3], and was formally developed in [10], with methods for constructing suitable Lyapunov functions discussed in [9]. The concepts introduced in both papers were integrated and elaborated upon in the monograph [11]. In contrast, a modern contribution by [20] proposes a systematic procedure for constructing Lyapunov functions specifically for second-order linear stochastic stationary systems.

The choice of a Lyapunov function is critical, as an inappropriate form may yield misleading or inconsistent results. Its construction demands careful consideration of the system's specific structure. However, no general method exists for constructing Lyapunov functions in either deterministic or stochastic contexts.

In deterministic systems, Lyapunov functions are often derived by working backward from the total time derivative, resulting in a first-order partial differential equation in spatial variables. Such equations are commonly solvable via methods like the d'Alembert's method of characteristics (e.g., [12]), providing viable Lyapunov candidates. This approach, however, does not directly extend to stochastic systems, where the relationship between the Lyapunov function and its evolution differs fundamentally.

In the stochastic domain, the classical total derivative is replaced by the adjoint Fokker-Planck (FP) operator $\mathbf{L}\{\cdot\}$, which accounts for the random function increment with respect to both time and space variables, known as the Wiener stochastic differential. This substitution reflects the fact that the magnitude of the stochastic increment $\|\Delta\mathbf{u}\|$ typically scales as $\sqrt{\Delta t}$, as discussed in [4, 19]. For a rigorous treatment of this framework, see [1, 7].

Given these challenges, alternative strategies are often required (e.g., [11]). One such approach leverages first integrals of the corresponding deterministic system. A classical example is the use of total energy—a conserved quantity—as a basis for constructing a Lyapunov function, an idea originally proposed by Chetaev in the deterministic setting [5]. This work adopts this approach to construct Lyapunov functions for systems with polynomial nonlinearities subjected to both Gaussian parametric and additive white noise.

This paper builds on presentations given at the 2022 and 2023 Computational Mechanics conferences and the 2022 Engineering Mechanics conference. It extends the previously introduced concepts in light of the valuable feedback and discussions that arose during these events.

The paper is organized as follows: Following this introduction, Section 2 provides general remarks on stochastic stability. Section 3 outlines the construction of Lyapunov functions based on first integrals of the underlying physical system. Section 4 illustrates this approach using a single-degree-of-freedom aeroelastic oscillator with Rayleigh- and van der Pol-type damping. Section 5 extends the analysis to a two-degree-of-freedom spherical pendulum. Section 6 presents a three-degree-of-freedom example involving a gyroscope, modeled as a perturbed force-free motion of a symmetric top. The final section summarizes the conclusions.

2. Stochastic stability and Lyapunov functions

A general stochastic system can be described in the following form:

$$\dot{u}_i = f_i(t,\mathbf{u}) + \sum_{k=1}^{n} w_{ik}(t) h_{ik}(\mathbf{u}). \qquad \mathbf{u}(t_0) = \mathbf{u}_0\,, \tag{1}$$

or, when the deterministic part $f_i(t,\mathbf{u})$ is separable in t and $\mathbf{u}$, as

$$\begin{aligned} \dot{u}_i &= \sum_{k=1}^{n} (A_{ik}(t) + w_{ik}(t)) h_{ik}(\mathbf{u})\,, \\ f_i(t,\mathbf{u}) &= \sum_{i=1}^{n} A_{ik}(t) h_{ik}(\mathbf{u})\,. \end{aligned} \tag{2}$$

Here, $h_{ik}(\mathbf{u})$ are the continuous diffusion-drift coupling functions, equations (1)–(2) represent an element-wise (Hadamard-style) weighting of $\mathbf{u} = \{u_i\}^T$, $i = 1,\ldots,n$, and $\dot{u}_i$ denotes the time derivative of u_i.

According to [4], the adjoint Fokker-Planck (FP) operator, also known as the *Kolmogorov backward operator*, is given by

$$\mathbf{L}\{\lambda(t,\mathbf{u})\} = \frac{\partial\lambda(t,\mathbf{u})}{\partial t} + \sum_{i=1}^{n} \kappa_i \frac{\partial\lambda(t,\mathbf{u})}{\partial u_i} + \frac{1}{2}\sum_{i,j=1}^{n} \kappa_{ij} \frac{\partial^2\lambda(t,\mathbf{u})}{\partial u_i \partial u_j}, \tag{3}$$

where κ_i and κ_{ij} represent the drift and diffusion coefficients, respectively, of the n-dimensional Markov process $\mathbf{u}(t)$ given by

$$\kappa_i = f_i(t,\mathbf{u}) + \frac{1}{2}\sum_{k,l,p=1}^{n} \frac{\partial h_{ik}(\mathbf{u})}{\partial u_p}\, h_{ip}(\mathbf{u}) s_{iklp}\,, \tag{4}$$

$$\kappa_{ij} = \sum_{k,l=1}^{n} h_{ik}(\mathbf{u}) h_{jl}(\mathbf{u}) s_{ikjl}\,. \tag{5}$$

In (1)–(5), the variables and parameters denote the following: $\lambda(t,\mathbf{u})$ is a candidate Lyapunov function, $A_{ik}(t)$ are the nominal (deterministic) system coefficients, possibly represented by $f_i(t,\mathbf{u})$ in (1), $w_{ik}(t)$ are zero-mean Gaussian white noise processes with cross-intensities s_{ikjl}, satisfying $\mathbf{E}\{w_{ik}(t)\, w_{jl}(t')\} = s_{ikjl}\,\delta(t-t')$, and $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation operator.

Note that the second term in (4) is specific to the Stratonovich interpretation of stochastic differential equations. In contrast, under Itô's interpretation—commonly used for systems driven by continuous processes—this term vanishes; see [19]. However, in scenarios where the Stratonovich model is appropriate, this correction term can play a critical role in determining the stability conditions.

The stability of the system described by (2) in the presence of random excitation is governed by the structure of the matrix $\mathbf{h}(\mathbf{u}) = (h_{ij}(\mathbf{u}))$ and the correlation properties of the noise processes $w_{ij}(t)$. For example, if $\mathbf{h}(\mathbf{u})$ is a square diagonal matrix and the noise processes are independent, then the parametric noise has a purely destabilizing effect. However, it is also possible to construct $\mathbf{h}(\mathbf{u})$ such that the influence of random excitation contributes positively to the system's stability.

The analysis of stochastic stability via Lyapunov functions closely relates to the deterministic case. Let $\lambda(t, \mathbf{u})$ be a suitable candidate function satisfying the following conditions:

(a) $\lambda(t, \mathbf{u})$ is positive definite in the Lyapunov sense, that is:

$$\begin{aligned} &\lambda(t, 0) = 0 \qquad &&\text{for all } \ t > 0\,, \quad \text{and} \\ &\lambda(t, \mathbf{u}) > \chi(\mathbf{u}) &&\text{for all } \mathbf{u} \neq \mathbf{0},\ t \geq 0\,, \end{aligned} \tag{6}$$

where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a continuous positive definite function, i.e., $\chi(\mathbf{0}) = 0$ and $\chi(\mathbf{u}) > 0$ for all $\mathbf{u} \neq \mathbf{0}$.

(b) $\lambda(t, \mathbf{u})$ is continuous, with a continuous first derivative in time and continuous second derivatives with respect to spatial variables. Consequently, the function

$$\psi(t, \mathbf{u}) = \mathbf{L}\{\lambda(t, \mathbf{u})\} \tag{7}$$

should also be continuous.

If, after substituting $\dot{\mathbf{u}}$ using (2), the function $\psi(t, \mathbf{u})$ is negative throughout a domain $\boldsymbol{\Omega}$ and vanishes or is undefined at the origin, then $\lambda(t, \mathbf{u})$ qualifies as a Lyapunov function. In such a case, for any initial condition with $\|\mathbf{u}_0\| \neq 0$, the function $\lambda(t, \mathbf{u})$ monotonically decreases as $t \to \infty$, while tending to zero in all spatial coordinates $\mathbf{u}$. This behavior implies that the trivial solution of the system given by (2) is stable in probability.

3. First integrals and the Lyapunov function

Consider a system whose equations of motion admit a first integral of the form

$$J_0(\mathbf{u}) = C_0 = \text{const}, \quad \mathbf{u} = (u_1, \ldots, u_n)^T, \tag{8}$$

where the difference $J_0(\mathbf{u}) - J_0(\mathbf{0})$ defines a positive definite function of the phase variables $\mathbf{u}$. In such cases, it is natural to select

$$\lambda(\mathbf{u}) = J_0(\mathbf{u}) - J_0(\mathbf{0}) \tag{9}$$

as a candidate Lyapunov function.

In purely conservative systems, the total mechanical energy can serve as a Lyapunov function. When nonconservative forces are introduced, stability is influenced by the type of these forces—dissipative forces that dissipate energy and gyroscopic forces that preserve energy but affect the system's dynamic coupling.

The equations of motion in Lagrangian form are given by

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_k} - \frac{\partial T}{\partial u_k} + \frac{\partial \Pi}{\partial u_k} = D_k + \Gamma_k, \tag{10}$$

where T is the kinetic energy, Π is the potential energy, and D_k represents dissipative forces. Γ_k denotes generalized gyroscopic (or workless) forces, which satisfy

$$\sum_k \Gamma_k \dot{u}_k = 0\,, \tag{11}$$

so they contribute no net work and, therefore, do not change the mechanical energy.

This notation separates the symmetric (inertial) part from the skew-symmetric (gyroscopic) part of the operator. Only the symmetric part contributes to the quadratic form associated with the energy function, whereas the gyroscopic part preserves energy and influences the system's stability solely through its coupling structure.

With this convention, and assuming that D_k and Γ_k are Euler-homogeneous functions of the phase coordinates, the total energy balance can be written as

$$\frac{d}{dt}\left(T+\Pi\right) = -\sum m\,\Psi_m, \tag{12}$$

where m denotes the degree of homogeneity of the function Ψ_m. This form highlights how the rate of change of total energy is governed by the structure of the nonconservative forces acting on the system.

Equation (12) suggests the existence of a first integral corresponding to the total mechanical energy. In the deterministic setting, if $T+\Pi$ is selected as the Lyapunov function, the right-hand side of (12) characterizes the system's stability domain, as follows from the Lyapunov's direct method.

When the system is perturbed by Gaussian parametric white noise, a modified formulation analogous to (12) can be obtained

$$\mathbf{L}\left\{T+\Pi\right\} = -\sum m\,\Theta_m\,, \tag{13}$$

where $\mathbf{L}\{\cdot\}$ denotes the adjoint Fokker-Planck operator, and Θ_m is a homogeneous function of the phase coordinates and noise intensities. Stability assessment proceeds analogously to the deterministic case.

Beyond (8), the system may possess additional first integrals

$$J_1(\mathbf{u}) = C_1,\ \ldots\ ,\ J_s(\mathbf{u}) = C_s, \tag{14}$$

though these integrals may not yield positive definite functions when reformulated as in (9). While it is possible to retain the Lyapunov function in the form of (9), neglecting the contribution of additional integrals in (14) may lead to overly conservative or inconclusive results due to a significant gap between necessary and sufficient stability conditions.

A more comprehensive Lyapunov function may be constructed as a linear combination of the first integrals and their functions, for instance,

$$\lambda(\mathbf{u}) = \sum_{i=1}^{s} a_i\big(J_i(\mathbf{u}) - J_i(\mathbf{0})\big) + b_i\big(J_i^2(\mathbf{u}) - J_i^2(\mathbf{0})\big), \tag{15}$$

where a_i and b_i are scalar coefficients chosen to ensure that $\lambda(\mathbf{u})$ is positive definite.

First integrals of the type given in (14) are commonly associated with systems possessing cyclic coordinates. In such cases, equation (10) reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_k} = D_k + \Gamma_k, \tag{16}$$

leading to first integrals of the form

$$\frac{\partial T}{\partial \dot{u}_k} = -\sum m\,\Psi_m. \tag{17}$$

This reduction enables analysis of the structure of generalized impulses. In the absence of dissipative and gyroscopic forces (i.e., $D_k = \Gamma_k = 0$), the generalized impulse remains conserved. Extension of this concept to the stochastic domain is evident.

In conclusion, several additional remarks can be made regarding the general construction principle: (i) one of the coefficients a_i or b_i may be fixed arbitrarily—for example, by setting $a_1 = 1$; (ii) in many cases, the linear part of (15) alone provides a sufficient Lyapunov candidate; (iii) in certain situations, first integrals can be identified directly from physical considerations, even without explicit knowledge of the governing equations. It should be noted, however, that the construction of a Lyapunov function—even when based on first integrals—retains a degree of subjectivity. Consequently, modifications to (15) beyond the prescribed algorithm are permissible, provided that the resulting function preserves the required properties of a Lyapunov function.

4. Single-degree-of-freedom aeroelastic system

The motion of a prismatic body transverse to an airflow results from aeroelastic interaction between the body and the surrounding fluid, as illustrated in Fig. 1. The associated pressure fluctuations in the airflow are predominantly random. When modeling galloping using a single-degree-of-freedom (SDOF) system, the dynamics can be described by a nonlinear differential equation, as presented in [8, 14, 21],

$$M\ddot{u}(t) + F_{\text{dam}}(u, \dot{u}) + Cu(t) = \varphi_M(t)\,, \tag{18}$$

where M and C denote the mass and stiffness coefficients, respectively, and $F_{\text{dam}}(u, \dot{u})$ represents the nonlinear damping force. In the absence of external forcing, the excitation is embedded within the damping term. Variations in the lift force due to changes in the angle of attack may lead to self-excitation. Additionally, random fluctuations in air pressure act as stochastic perturbations of the damping mechanism and stiffness coefficient. This effect may be formally written as $\varphi_M(t) = -M\,(w_1 u - w_2 \dot{u})$, where w_1, w_2 represent the perturbations.

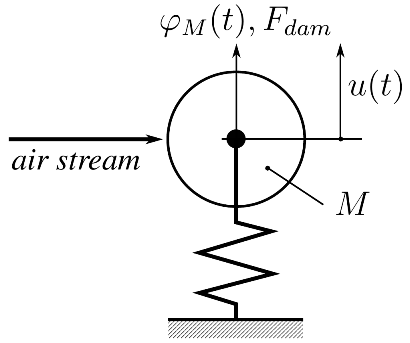


Fig. 1. Schematic of an SDOF system exposed to transverse air flow

Several forms of $F_{\text{dam}}(u, \dot{u})$ have been proposed. The most common lead to two canonical types:

(i) *Rayleigh-type damping:*

$$F_{\text{dam}}(u, \dot{u}) = 2M \left[\left(\omega_{bc} + \frac{1}{2} \delta^2 \dot{u}^2(t) \right) - \omega_{ba} \right] \dot{u}(t) \,, \tag{19}$$

(ii) *van der Pol-type damping:*

$$F_{\text{dam}}(u, \dot{u}) = 2M \left[\left(\omega_{bc} + \frac{1}{2} \gamma^2 u^2(t) \right) - \omega_{ba} \right] \dot{u}(t) \,. \tag{20}$$

In both expressions, ω_{bc} corresponds to the classical viscous damping coefficient, while ω_{ba} accounts for aerodynamic effects due to lift variation. Their difference $\omega_b = \omega_{ba} - \omega_{bc}$ characterizes the net contribution of aerodynamic destabilization. The parameters δ and γ define the intensity of the nonlinear damping terms.

These damping components are subject to random fluctuations due to airflow irregularities, modeled as Gaussian white noise processes $w_1(t)$ and $w_2(t)$, with respective intensities s_{11}, s_{22}, and cross-correlation intensity s_{12}.

Introducing the phase variables $\mathbf{u} = (u_1, u_2)^T = (u, \dot{u})^T$, the system governed by (18) can be reformulated in normal form as follows:

(i) *Rayleigh-type system:*

$$\begin{aligned} \dot{u}_1 &= u_2 \,, \\ \dot{u}_2 &= \left(2\omega_b - \delta^2 u_2^2 + w_2 \right) u_2 - (\omega_0^2 + w_1) u_1 \,, \end{aligned} \tag{21}$$

(ii) *van der Pol-type system:*

$$\begin{aligned} \dot{u}_1 &= u_2 \,, \\ \dot{u}_2 &= \left(2\omega_b - \gamma^2 u_1^2 + w_2 \right) u_2 - (\omega_0^2 + w_1) u_1 \,, \end{aligned} \tag{22}$$

where $\omega_0^2 = C/M$ denotes the squared natural frequency of the undisturbed linear system.

Both damping models possess a first integral of motion corresponding to total energy. Therefore, an appropriate deterministic Lyapunov function can be defined, as in [15, 16, 18], by

(i) *for Rayleigh-type damping:*

$$\lambda(t, \mathbf{u}) = \frac{1}{2} u_2^2 + \frac{1}{2} \omega_0^2 u_1^2 \,, \tag{23}$$

(ii) *for van der Pol-type damping:*

$$\lambda(t, \mathbf{u}) = \frac{1}{2} \big(u_2 - G(u_1) \big)^2 + \frac{1}{2} \omega_0^2 u_1^2 \,, \tag{24}$$

where the function $G(u_1)$ in (24) incorporates the nonlinear damping contribution

$$G(u_1) = \int_0^{u_1} \left(2\omega_b - \gamma^2 \zeta^2 \right) \, \mathrm{d}\zeta = 2\omega_b u_1 - \frac{1}{3} \gamma^2 u_1^3 \,. \tag{25}$$

Both equations (23) and (24) define positive definite functions. In accordance with the notation introduced in (2), (4) and (5), the coefficients κ_i and κ_{ij} take the following forms:

$$\text{(i)} \qquad \begin{aligned} \kappa_1 &= u_2\,, \\ \kappa_2 &= (2\omega_b - \delta^2 u_2^2)u_2 - \omega_0^2 u_1\,, \\ \kappa_{11} &= \kappa_{12} = \kappa_{21} = 0\,, \\ \kappa_{22} &= u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22}\,, \end{aligned} \tag{26}$$

$$\text{(ii)} \qquad \begin{aligned} \kappa_1 &= u_2\,, \\ \kappa_2 &= (2\omega_b - \gamma^2 u_1^2)u_2 - \omega_0^2 u_1\,, \\ \kappa_{11} &= \kappa_{12} = \kappa_{21} = 0\,, \\ \kappa_{22} &= u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22}\,, \end{aligned} \tag{27}$$

where, with respect to (2), $n = 2, m = 3, h_{12} = h_{22} = u_2, h_{21} = u_1$; $h_{23} = u_2^3$ or $u_1^2 u_2$ for cases (i) and (ii), respectively. The stochastic inputs are defined as $w_{21} = -w_1, w_{22} = w_2$; $s_{ikjl} = s_{kl}$ for $i = 2$ or $j = 2$, and $s_{ikjl} = 0$ whenever any of the following holds: $i = 1$, $j = 1$, $k = 3$ or $l = 3$.

Denoting $\partial_x = \partial/\partial x$, the derivatives of the Lyapunov function $\lambda(t, \mathbf{u})$ with respect to the phase variables, as required in (3), are given by

$$\text{(i)} \qquad \begin{aligned} \partial_{u_1}\lambda(t,\mathbf{u}) &= \omega_0^2 u_1\,, \\ \partial_{u_2}\lambda(t,\mathbf{u}) &= u_2\,, \\ \partial_{u_2^2}\lambda(t,\mathbf{u}) &= 1\,, \end{aligned} \tag{28}$$

$$\text{(ii)} \qquad \begin{aligned} \partial_{u_1}\lambda(t,\mathbf{u}) &= \left(u_2 - (2\omega_b - 1/3\,\gamma^2 u_1^2)u_1\right)(\gamma^2 u_1^2 - 2\omega_b) + \omega_0^2 u_1\,, \\ \partial_{u_2}\lambda(t,\mathbf{u}) &= \left(u_2 - (2\omega_b - 1/3\,\gamma^2 u_1^2)u_1\right)\,, \\ \partial_{u_2^2}\lambda(t,\mathbf{u}) &= 1\,, \end{aligned} \tag{29}$$

Finally, combining the partial expressions from (26)–(29), the application of the adjoint FP operator $\mathbf{L}\{\cdot\}$ to the Lyapunov function yields

$$\mathbf{L}\{\lambda(t,\mathbf{u})\} =$$

$$\text{(i)} \qquad \psi(t,\mathbf{u}) = u_2^2\left(2\omega_b - \delta^2 u_2^2\right) + u_1^2 s_{11} + u_1 u_2(s_{12} + s_{21}) + u_2^2 s_{22}\,, \tag{30}$$

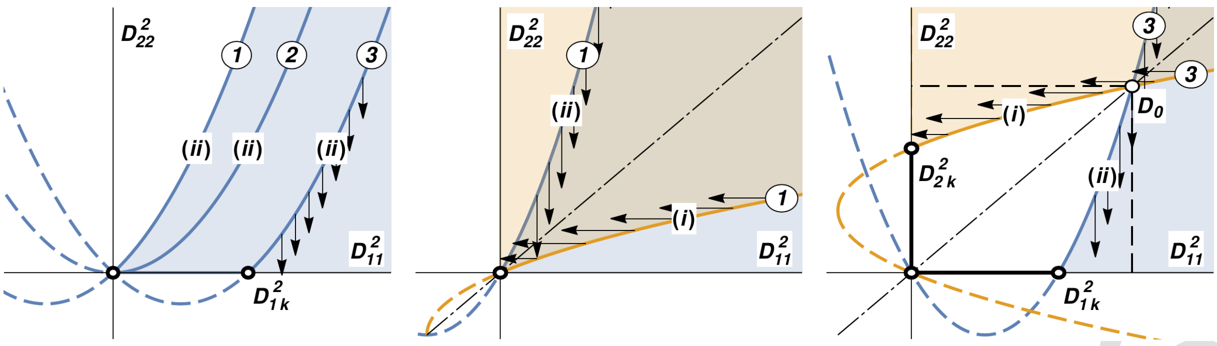
$$\text{(ii)} \qquad \psi(t,\mathbf{u}) = \omega_0^2 u_1^2\left(2\omega_b - 1/3\,\gamma^2 u_1^2\right) + u_1^2 s_{11} + u_1 u_2(s_{12} + s_{21}) + u_2^2 s_{22}\,. \tag{31}$$

When the noises w_1 and w_2 are independent (i.e., $s_{12} = s_{21} = 0$), it follows from the structure of (30)–(31) that both noise terms have a destabilizing effect compared to the deterministic case. The condition $\psi(t, \mathbf{u}) = 0$ can be employed to estimate the stability boundaries of the original system. Since the model includes only symmetric, independent parametric noise sources without external excitation, the system's response is likewise symmetric. As a result, the processes u_1 and u_2 can be treated as centered.

If the process u_1 can be approximately considered Gaussian, the relation $D_{ii}^4 = 3(D_{ii}^2)^2$ holds, where D_{ii}^2 and D_{ii}^4 denote the second and fourth central moments of u_i, respectively. This approximation is reasonable, particularly for small amplitudes when the systems governed by (19) and (20) behave approximately linearly. In such cases, as shown in [18], the estimated stability boundary in the domain $D_{11}^2, D_{22}^2 > 0$ takes the form:

$$\text{(i)} \qquad -3\delta^2(D_{22}^2)^2 + (2\omega_b + s_{22})D_{22}^2 + s_{11}D_{11}^2 = 0\,, \tag{32}$$

$$\text{(ii)} \qquad -\omega_0^2\gamma^2(D_{11}^2)^2 + (2\omega_0^2\omega_b + s_{11})D_{11}^2 + s_{22}D_{22}^2 = 0\,. \tag{33}$$



(a) (b) (c)

Fig. 2. (a) Stability domains for the van der Pol system (case (ii)); flow speeds: ① – subcritical, ② – critical, ③ – supercritical. (b)–(c) Comparison of stability domains for the Rayleigh (parabola with horizontal axis, yellow, case (i)) and van der Pol (parabola with vertical axis, blue, case (ii)) damping models for subcritical (b) and supercritical (c) flow speeds

The curves defined by (32)–(33) are parabolas passing through the origin. Their axes are (i) horizontal for (32) and (ii) vertical for (33). The stable region corresponds to the domain where $\psi(t,\mathbf{u}) < 0$, i.e., the region between the respective parabola and either (i) the positive vertical axis of D_{22}^2 or (ii) the positive horizontal axis of D_{11}^2. These scenarios give rise to three distinct cases, illustrated in Fig. 2a for the van der Pol system in case (ii).

In all graphs in Fig. 2, the shaded areas together with the arrows indicate the stability domains, i.e., the regions where conditions (i) or (ii) are satisfied. In plot (a), the three curves ①–③ defined by (33) represent subcritical, critical, and supercritical flow velocities, respectively. For each curve, the stability region occupies the area between the parabola and the positive part of the horizontal axis. The three cases differ in whether the stability domain includes the origin (subcritical), touches it (critical), or is displaced away from it (supercritical).

If the vertex of the parabola lies to the left of the D_{22}^2 axis in case (i), or below the D_{11}^2 axis in case (ii), the stability domain extends to the origin (cf. curve ① in Fig. 2a and both curves in Fig. 2b.). These conditions are satisfied when

$$\text{(i)} \qquad 2\omega_b + s_{22} < 0\,, \tag{34}$$

$$\text{(ii)} \qquad 2\omega_0^2\omega_b + s_{11} < 0\,. \tag{35}$$

If the vertex coincides with the origin—that is, if the inequalities in (34)–(35) become equalities—the system is marginally stable near the origin. This situation corresponds to curve ② in Fig. 2a. In this case, it is possible for $\psi(t,\mathbf{u})$ to attain zero or negative values without requiring positive values of D_{11}^2 or D_{22}^2. The system is thus poised at the boundary between unconditional stability and the onset of nonlinear, amplitude-dependent stabilization.

If either expression in (34)–(35) is positive, the vertex of the corresponding parabola lies above the D_{11}^2 axis or to the right of the D_{22}^2 axis. This configuration is depicted by curve ③ in Fig. 2a and by both curves in Fig. 2c. In such cases, secondary stability arises due to the influence of the nonlinear damping terms, and the system exhibits bounded vibration within a nonzero band. The onset of stability occurs at points D_{2k}^2 and D_{1k}^2, given by

$$\text{(i)} \qquad D_{2k}^2 = \frac{2\omega_b + s_{22}}{3\delta^2}\,, \tag{36}$$

$$\text{(ii)} \qquad D_{1k}^2 = \frac{2\omega_0^2\omega_b + s_{11}}{\omega_0^2\gamma^2}\,, \tag{37}$$

and are adjacent to the vertical axis D_{22}^2 or the horizontal axis D_{11}^2, respectively. These threshold values mark the minimum amplitude of effective nonlinear damping required to counteract the destabilizing influence of the flow.

When the airflow velocity—or equivalently, the parameter ω_b—is sufficiently low such that the stability domains in both cases extend to the origin, the system remains stable, and the displacement perturbations due to parametric noise are negligible. In this regime, the system's behavior is largely independent of the damping model employed. Both Rayleigh- and van der Pol-type nonlinearities produce nearly identical stability characteristics in this low-velocity range.

In the supercritical regime, a common instability domain emerges for both models, and the system's response becomes highly sensitive to the specific form of the damping characteristics; see Fig. 2c. As the airflow velocity increases, the stability boundary shifts along the diagonal of the first quadrant. Beyond a certain velocity threshold, denoted D_0 in Fig. 2c, the system enters a nonlinear stability domain governed by the damping nonlinearity. This stability domain is shared by both models and shows decreasing sensitivity to the exact form of the nonlinear damping function. Similar behavior has also been reported in two-degree-of-freedom systems; see [13,17]. Thus, at high flow velocities the dominance of nonlinear damping leads to a unified stabilization mechanism, regardless of the specific damping formulation.

5. Two-degree-of-freedom spherical pendulum

The motion of a spherical pendulum rotating about a vertical axis serves as a more complex example for constructing a Lyapunov function. The system dynamics are described using the coordinates φ (the azimuthal angle around the vertical axis) and ξ (the angle between the vertical axis and the pendulum arm). In the deterministic stationary state, the pendulum undergoes uniform circular motion in the horizontal plane, as illustrated in Fig. 3 and discussed in [6]. However, in the presence of small random perturbations, the motion is altered.

The perturbed motion can be described by the expressions

$$\xi = \alpha + u_1, \qquad \dot{\xi} = u_2, \qquad \dot{\varphi} = \omega + u_3\,, \tag{38}$$

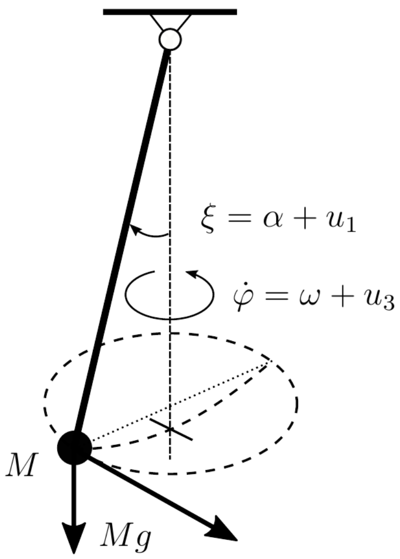


Fig. 3. Outline of a two-degree-of-freedom pendulum

where α denotes the nominal angle between the pendulum suspension and the vertical in the unperturbed (deterministic) state. The relation $\omega^2 l \cos\alpha = g$ defines the connection between the angular velocity ω, pendulum length l, and gravitational acceleration g.

The kinetic and potential energies, T and Π, are given by

$$T = \frac{1}{2} M l^2 (\dot{\xi}^2 + \dot{\varphi}^2 \sin^2 \xi)\,, \tag{39}$$

$$\Pi = -Mgl\, \cos\xi\,. \tag{40}$$

The kinetic energy T depends on the velocity $\dot{\varphi}$ but not on the coordinate φ itself. Since $\partial\Pi/\partial\varphi = 0$, the coordinate φ is cyclic, and no force acts directly along it. Moreover, because gravity is a potential force, the system possesses two first integrals

$$T + \Pi = \frac{1}{2} M l^2 (\dot{\xi}^2 + \dot{\varphi}^2 \sin^2 \xi) - Mgl\, \cos\xi\,, \tag{41}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = M l^2 \dot{\varphi}\, \sin^2 \xi\,. \tag{42}$$

Equation (41) represents the total mechanical energy and corresponds to (12). Equation (42) corresponds to the integral form in (17) and expresses conservation of angular momentum about the vertical axis. These integrals, derived from general dynamical principles, allow for system characterization without requiring the explicit derivation of the governing differential equations.

Using (41)–(41), a Lyapunov function can be constructed in the form of (15) to analyze the stability of the pendulum's motion. In this particular case, it is sufficient to consider only the linear part of (15) by setting $b_1 = b_2 = 0$. This leads to the form

$$\lambda(u_1, u_2, u_3) = a_1\big(J_1(\mathbf{u}) - J_1(\mathbf{0})\big) + a_2\big(J_2(\mathbf{u}) - J_2(\mathbf{0})\big), \tag{43}$$

where J_1 and J_2 are derived by substituting (38) into (41)–(41), i.e.,

$$\begin{aligned} 2\left(Ml^2\right)^{-1} J_1(\mathbf{u}) &= (\omega + u_3)^2 \sin^2(\alpha + u_1) - \frac{2g}{l}\cos(\alpha + u_1) + u_2^2\,, \\ \left(Ml^2\right)^{-1} J_2(\mathbf{u}) &= (\omega + u_3) \sin^2(\alpha + u_1)\,. \end{aligned} \tag{44}$$

The coefficients a_1 and a_2 can be selected as

$$a_1 = 2\left(Ml^2\right)^{-1}\,, \qquad a_2 = a\left(Ml^2\right)^{-1}\,, \tag{45}$$

where the parameter a is to be determined to ensure that the Lyapunov function λ is positive definite.

Substituting (44) into (43) yields

$$\begin{aligned} \lambda(u_1, u_2, u_3) = u_2^2 &+ (\omega + u_3)^2 \sin^2(\alpha + u_1) - \frac{2g}{l}\left[\cos(\alpha + u_1) - \cos\alpha\right] \\ &+ \left[(\omega + u_3)\sin^2(\alpha + u_1) - \omega \sin^2\alpha\right] a - \omega^2 \sin^2\alpha. \end{aligned} \tag{46}$$

Assuming small perturbations u_1, u_2, u_3, equation (46) can be expanded using a Taylor series. The resulting approximation is

$$\begin{aligned} \lambda(u_1, u_2, u_3) = u_1^2 \omega &\left[(a + \omega)\cos 2\alpha + \omega \cos^2\alpha\right] + u_2^2 + u_3^2 \sin^2\alpha \\ &+ (2u_1\omega\cos\alpha + u_3 \sin\alpha + 2u_3 u_1 \cos\alpha)\,(a + 2\omega)\sin\alpha + \ldots \end{aligned} \tag{47}$$

Here, the relation $g = \omega^2 l \cos\alpha$ has been applied, which reflects the equilibrium condition where the centrifugal force resulting from rotation balances the component of gravitational force along the pendulum arm.

To ensure that $\lambda(u_1, u_2, u_3)$ is positive definite, the linear terms in u_1, u_2, and u_3 must be eliminated. This condition is satisfied by choosing $a = -2\omega$. Substituting this value yields the simplified Lyapunov function

$$\lambda(u_1, u_2, u_3) = (u_1^2\omega^2 + u_3^2)\sin^2\alpha + u_2^2 + \ldots \tag{48}$$

Returning to the system under consideration, the equations of motion can be derived using (39)–(40) in conjunction with the Lagrange equations, within the scale of perturbations. They take the following form:

$$\ddot{\xi} - \dot{\varphi}^2 \sin\xi\cos\xi + \frac{g}{l}\sin\xi = \mu l(\xi - \alpha)w(t), \tag{49}$$

$$\ddot{\varphi} + 2\dot{\varphi}\dot{\xi}\cot\xi = 0. \tag{50}$$

To introduce parametric excitation, equation (49) includes the influence of white noise $w(t)$, which is proportional to the deviation from the unperturbed angle α; $\mu\,[\mathrm{s}^{-2}]$ denotes the coupling constant. Substituting $\xi, \dot{\xi}, \dot{\varphi}$ from (38) into (49)–(50), the system can be reformulated in normal form in terms of u_1, u_2, u_3 as

$$\begin{aligned}
\dot{u}_1 &= u_2\,, \\
\dot{u}_2 &= -\omega^2\cos\alpha\sin(\alpha + u_1) + \frac{1}{2}(\omega + u_3)^2\sin 2(\alpha + u_1) + \mu l\, u_1 w(t)\,, \\
\dot{u}_3 &= -2u_2(\omega + u_3)\,\cot(\alpha + u_1)\,.
\end{aligned} \tag{51}$$

The linearized form of the system is given by

$$\begin{aligned}
\dot{u}_1 &= u_2\,, \\
\dot{u}_2 &= -\omega^2 u_1\sin^2\alpha + \omega\, u_3\sin 2\alpha + \mu l\, u_1 w(t)\,, \\
\dot{u}_3 &= -2u_2\,\omega\cot\alpha\,.
\end{aligned} \tag{52}$$

In order to formulate the Lyapunov function using (3), the drift and diffusion coefficients κ_i and κ_{ij} corresponding to (52) are given by

$$\begin{aligned}
\kappa_1 &= u_2\,, \\
\kappa_2 &= -u_1\omega^2\sin^2\alpha + u_3\omega\sin 2\alpha\,, \\
\kappa_3 &= -2u_2\omega\cot\alpha\,.
\end{aligned}
\qquad
\kappa_{ij} = \begin{cases} \mu^2 s_{ww} u_1^2 l^2\,, & i = j = 2\,, \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

In the case of the spherical pendulum described by (51), the function $\psi(\mathbf{u})$ takes the form

$$\begin{aligned}
\psi(\mathbf{u}) = {} & 2u_1u_2\omega^2\,\sin^2\alpha + u_2\left[(\omega + u_3)^2\sin 2(\alpha + u_1) - 2\omega^2\cos\alpha\sin(\alpha + u_1)\right] \\
& - 4u_2u_3(\omega + u_3)\sin^2\alpha\,\cot(\alpha + u_1) + u_1^2(\mu l)^2 s_{ww}\,.
\end{aligned} \tag{54}$$

Alternatively, when expanding and retaining terms up to third or second order in the perturbation variables u_i, the function simplifies as follows:

$$\psi(\mathbf{u}) = -\left[\left(\frac{3}{2}\omega^2 u_1^2 + u_3^2\right) - 8\omega u_1 u_3\cos^2\alpha\right] u_2 + u_1^2(\mu l)^2 s_{ww}\,, \tag{55}$$

$$\psi(\mathbf{u}) = u_1^2(\mu l)^2 s_{ww}\,, \tag{56}$$

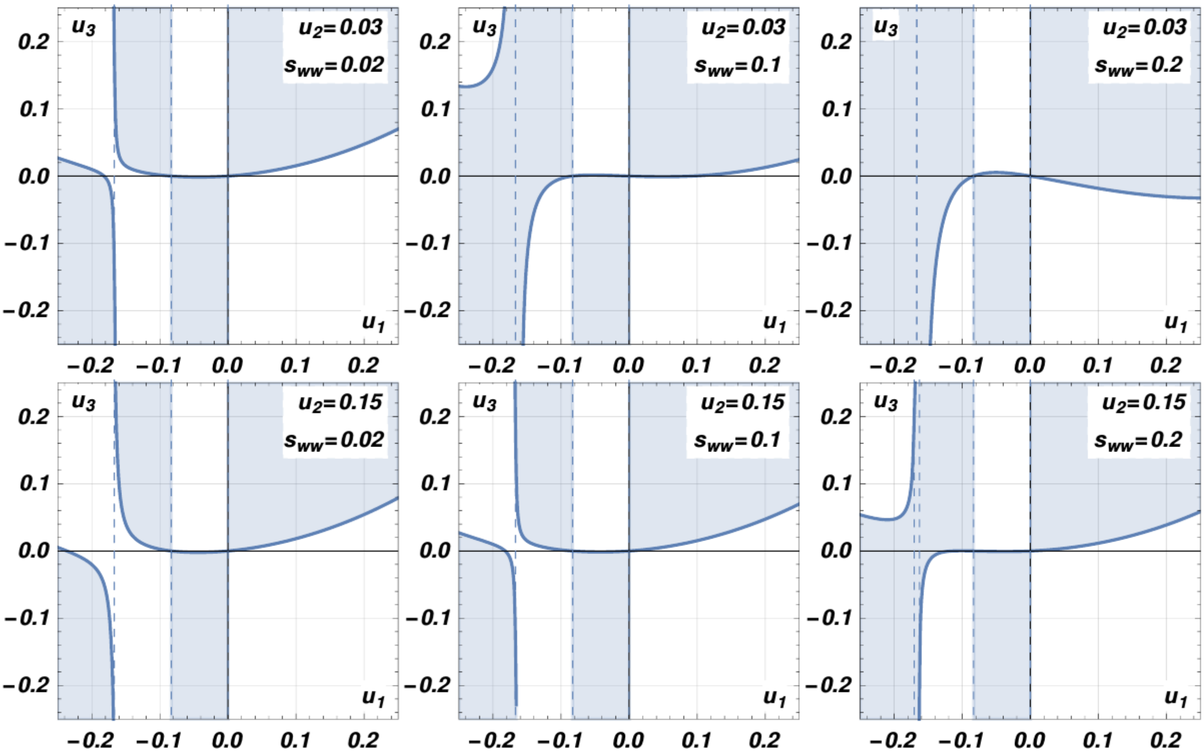


Fig. 4. Regions of positive (dark) and negative (bright) values of the Lyapunov function $\psi(\mathbf{u})$, as defined in (54), for various parameter configurations; parameters used: $l = 1\,\mathrm{m}$, $g = 9.81\,\mathrm{m\,s^{-2}}$, $\alpha = 1/12\,\mathrm{rad}$, and $s_{ww} = 0.02, 0.1, 0.2$

depending on whether cubic or only quadratic terms in u_i are retained.

The positive coefficient of s_{ww} in (54)–(56) clearly reflects the destabilizing influence of the noise $w(t)$. The overall stability of the system depends on the contributions of the remaining terms on the right-hand side of (54). When the Lyapunov function $\psi(\mathbf{u})$ is constructed based on the linearized version of the system, all terms vanish except the last one, due to the structure of the first integrals, see (56). This implies that, in the linearized approximation, the system is unstable.

However, stability can be achieved either by introducing dissipative forces or by carefully designing the matrix $\mathbf{h}(\mathbf{u})$ and tailoring the properties of the noise processes $w_i(t)$, depending on the physical nature of the system under investigation.

Fig. 4 presents a set of six contour plots of the Lyapunov function $\psi(\mathbf{u})$, as defined in (54), evaluated over the phase space u_1, u_3 for two values of $u_2 = 0.03,\ 0.15$. The results are arranged in two rows, each corresponding to a fixed value of u_2, which represents the deviation in the angular velocity of the pendulum suspension. Each row contains three subplots, corresponding to different levels of noise intensity, $s_{ww} = 0.02, 0.15, 0.3$, respectively. This allows for a direct comparison of how increasing stochastic excitation affects the qualitative structure of the Lyapunov function and, consequently, the system's stability.

Dark-shaded regions indicate domains where $\psi(\mathbf{u}) > 0$ signaling potential instability, whereas bright regions correspond to $\psi(\mathbf{u}) < 0$, suggesting local stability. As the noise intensity increases across each row, the regions of positive $\psi(\mathbf{u})$ expand, illustrating the destabilizing influence of stronger random part of parametric excitation. The analysis assumes fixed system parameters: $l = 1\,\mathrm{m}$, $g = 9.81\,\mathrm{m\,s^{-2}}$, $\mu = 1\,\mathrm{s^{-2}}$, and $\alpha = 1/12\,\mathrm{rad}$.

6. Gyroscope

This section considers a three-degree-of-freedom system representing the perturbed force-free motion of a gyroscope, as described in [22]. The symmetric top rotates about its symmetry axis z, which is aligned with a massless shaft hinged at the origin of a fixed coordinate system. The center of mass is located at a distance l above the hinge point. The primary rotational motion may be influenced by parasitic perturbations about the horizontal axes. The orientation of the gyroscope is described using a moving coordinate system (x, y, z), which is related to the inertial frame (x_0, y_0, z_0) via the Euler angles α and β, as illustrated in Fig. 5.

In the unperturbed state, the motion is purely rotational about the z-axis, and the values of the five dynamical variables reduce to

$$\alpha = 0\,, \quad \dot{\alpha} = 0\,, \quad \beta = 0\,, \quad \dot{\beta} = 0\,, \quad \dot{\varphi} = \omega\,. \tag{57}$$

The kinetic and potential energy of the gyroscopic system are given by

$$\begin{aligned} T &= \frac{1}{2} I_x(\dot{\alpha}^2 + \dot{\beta}^2 \cos^2 \alpha) + \frac{1}{2} I_z(\dot{\varphi} - \dot{\beta} \sin \alpha)^2, \\ \Pi &= Mgl\, \cos \alpha \cos \beta, \end{aligned} \tag{58}$$

where $I_x = I_y$ and I_z are the principal moments of inertia, M is the mass, and l is the vertical distance from the hinge to the center of mass.

The system admits three first integrals that characterize its deterministic dynamics and serve as the basis for constructing suitable Lyapunov functions in the presence of stochastic perturbations. The first is the total mechanical energy

$$T + \Pi = \frac{1}{2} I_x(\dot{\alpha}^2 + \dot{\beta}^2 \cos^2 \alpha) + \frac{1}{2} I_z(\dot{\varphi} - \dot{\beta} \sin \alpha)^2 + Mgl\, \cos \alpha \cos \beta = C_1\,. \tag{59}$$

Because the coordinate φ is cyclic, the second integral arises from conservation of angular momentum about the body's symmetry axis

$$\frac{\partial T}{\partial \dot{\varphi}} = I_z(\dot{\varphi} - \dot{\beta} \sin \alpha) = I_z\, C_2\,. \tag{60}$$

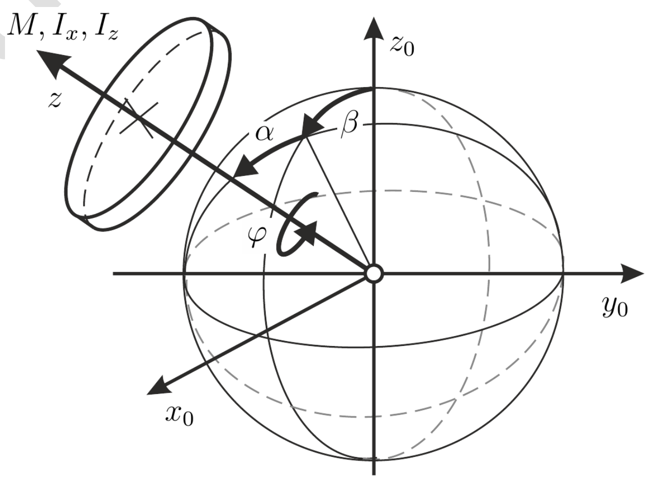


Fig. 5. Schematic of the gyroscope and its coordinate systems

The third integral corresponds to the projection of the angular momentum vector onto the fixed vertical axis z_0, and can be expressed as

$$I_x(-\dot{\alpha}\sin\beta + \dot{\beta}\cos\alpha\sin\alpha\cos\beta) + I_z(\dot{\varphi} - \dot{\beta}\sin\alpha)\cos\alpha\cos\beta = C_3\,. \tag{61}$$

Perturbations of the motion are introduced through the coordinates $\mathbf{u} = u_i$, $i = 1, \ldots, 5$, defined as follows

$$\alpha = u_1, \qquad \dot{\alpha} = u_2, \qquad \beta = u_3, \qquad \dot{\beta} = u_4, \qquad \dot{\varphi} = \omega + u_5.$$

The first integrals of the perturbed motion take the form

$$\begin{aligned}
J_1(\mathbf{u}) &= \frac{1}{2}I_x(u_2^2 + u_4^2\cos^2 u_1) + \frac{1}{2}I_z(\omega + u_5 - u_4\sin u_1)^2 + Mgl\,\cos u_1\cos u_3\,,\\
J_2(\mathbf{u}) &= \omega + u_5 - u_4\sin u_1\,,\\
J_3(\mathbf{u}) &= I_x(-u_2\sin u_3 + u_4\cos u_1\sin u_1\cos u_3) + I_z(\omega + u_5 - u_4\sin u_1)\cos u_1\cos u_3\,.
\end{aligned} \tag{62}$$

Since the second and third integrals are not positive definite, a Lyapunov function is constructed in the following form:

$$\lambda(\mathbf{u}) = a_1(J_1(\mathbf{u}) - J_1(\mathbf{0})) + a_2(J_2(\mathbf{u}) - J_2(\mathbf{0})) + a_3(J_3(\mathbf{u}) - J_3(\mathbf{0})), \quad a_1 = 1. \tag{63}$$

Substituting the expressions for the first integrals from (62) into (63), and expanding the trigonometric functions $\sin$ and $\cos$ in a Taylor series up to second order (assuming small perturbations), yields the following quadratic approximation:

$$\begin{aligned}
\lambda(\mathbf{u}) = &-\frac{1}{2}(Mgl + a_3\omega I_z)u_1^2 + \frac{1}{2}I_x u_2^2 - \frac{1}{2}(Mgl + a_3\omega I_z)u_3^2 + \frac{1}{2}I_x u_4^2 + \frac{1}{2}I_z u_5^2\\
&+ (\omega I_z + a_2 + a_3 I_z)u_5 - (\omega I_z + a_2 + a_3 I_z - a_3 I_x)u_1 u_4 - a_3 I_x u_2 u_3 + \ldots
\end{aligned} \tag{64}$$

To ensure that the Lyapunov function $\lambda(\mathbf{u})$ is positive definite, the coefficient of the linear term in u_5 must vanish. This requirement leads to the condition

$$\omega I_z + a_2 + a_3 I_z = 0\,. \tag{65}$$

With this condition imposed, equation (64) simplifies to

$$\lambda(\mathbf{u}) = \frac{1}{2}\delta u_1^2 + \frac{1}{2}I_x u_2^2 + \frac{1}{2}\delta u_3^2 + \frac{1}{2}I_x u_4^2 + \frac{1}{2}I_z u_5^2 + a_3 I_x u_1 u_4 - a_3 I_x u_2 u_3 + \ldots.\,, \tag{66}$$

$$\delta = -\,(Mgl + a_3\omega I_z)\,. \tag{67}$$

The function defined in (66) can be decomposed into the sum of three quadratic terms

$$\begin{aligned}
\lambda_1(\mathbf{u}) &= \frac{1}{2}\delta u_1^2 + a_3 I_x u_1 u_4 + \frac{1}{2}I_x u_4^2\,,\\
\lambda_2(\mathbf{u}) &= \frac{1}{2}\delta u_3^2 - a_3 I_x u_2 u_3 + \frac{1}{2}I_x u_2^2\,,\\
\lambda_3(\mathbf{u}) &= \frac{1}{2}I_z u_5^2\,.
\end{aligned} \tag{68}$$

The term $\lambda_3(\mathbf{u})$ is positive definite in the variable u_5. The remaining terms $\lambda_1(\mathbf{u})$ and $\lambda_2(\mathbf{u})$ have identical structure and are positive definite under certain conditions. According to Sylvester's criterion, each of these quadratic forms is positive definite for non-zero (u_1, u_4) and (u_2, u_3) if the following inequalities are satisfied:

$$\begin{aligned} \Delta_1 &= I_x > 0\,, \\ \Delta_2 &= \begin{vmatrix} \delta & a_3 I_x \\ a_3 I_x & I_x \end{vmatrix} = I_x(\delta - a_3^2 I_x) > 0\,. \end{aligned} \tag{69}$$

Substituting for δ from (67) leads to the following condition for the coefficient a_3:

$$-\frac{1}{2I_x}\left(-\omega I_z - \sqrt{\omega^2 I_z^2 - 4MglI_x}\right) < a_3 < -\frac{1}{2I_x}\left(-\omega I_z + \sqrt{\omega^2 I_z^2 - 4MglI_x}\right). \tag{70}$$

Thus, for a sufficiently large angular velocity ω, ensuring that the discriminant in (70) is positive, that is, for

$$|\omega| > \frac{2\sqrt{MglI_x}}{I_z}\,,$$

there exist real values of a_3 satisfying the condition in (70). In such cases, the function $\lambda(\mathbf{u})$, as defined by (64) and (65), is positive definite and can therefore be used as a Lyapunov function.

The subsequent analysis proceeds analogously to the previous cases. Since the system of equations of motion in normal form is derived from the kinetic and potential energies in (58), and stochastic perturbations are introduced through physically motivated noise models, the resulting formulation corresponds to the structure of (1). This permits the determination of the function $\psi(\mathbf{u})$ using the same methodology described in the previous sections.

Accordingly, the investigation of stochastic stability can begin with the deterministic characteristics of the system and focus on the contribution of the final term in the Fokker-Planck operator (3), particularly in terms of whether it leads to positive or negative values of $\psi(\mathbf{u})$.

7. Conclusions

The construction of Lyapunov functions based on first integrals has been shown to offer significant advantages over more general methods. In the case of linear systems, this approach yields results consistent with the classical Routh-Hurwitz stability criteria, providing both necessary and sufficient conditions for stability. For nonlinear systems, the analysis can be effectively grounded in the deterministic dynamics, as the impact of parametric stochastic perturbations can be treated as a distinct extension. This separation enables a clear and tractable analysis of the stochastic behavior and supports the development of models that account for the stabilizing or destabilizing effects of random parametric noise.

A major strength of this method lies in its ability to incorporate the physical and structural characteristics of the system directly into the form of the Lyapunov function. In many cases, it is not necessary to derive the full equations of motion explicitly; instead, stability properties can be inferred directly from the system's first integrals. When refining the Lyapunov function—typically constructed as a linear combination of these integrals—it is often necessary to impose constraints to ensure positive definiteness. These constraints frequently have direct physical interpretations, making them not only mathematically necessary but also practically meaningful in terms of system design and control.

Although the approach is not universally applicable to all classes of dynamical systems, it proves particularly effective for a wide range of nonlinear systems, including those considered in this study. The method is especially advantageous when first integrals are known or can be derived from symmetry or conservation laws. In such cases, it enables a rigorous and physically interpretable stability analysis, both in deterministic and stochastic settings.

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