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Bending of a nonlinear beam reposing on an unilateral foundation

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Abstract

This article is going to deal with bending of a nonlinear beam whose mathematical model was proposed by D. Y. Gao in (Gao, D. Y., Nonlinear elastic beam theory with application in contact problems and variational approaches, Mech. Research Communication, 23 (1) 1996). The model is based on the Euler-Bernoulli hypothesis and under assumption of nonzero lateral stress component enables moderately large deflections but with small strains. This is here extended by the unilateral Winkler foundation. The attribution unilateral means that the foundation is not connected with the beam. For this problem we demonstrate a mathematical formulation resulting from its natural decomposition which leads to a saddle-point problem with a proper Lagrangian. Next we are concerned with methods of solution for our problem by means of the finite element method as the paper (Gao, D. Y., Nonlinear elastic beam theory with application in contact problems and variational approaches, Mech. Research Communication, 23 (1) 1996) has no mention of it. The main alternatives are here the solution of a system of nonlinear nondifferentiable equations or finding of a saddle point through the use of the augmented Lagrangian method. This is illustrated by an example in the final part of the article.

Keywords: nonlinear beam, unilateral Winkler foundation, saddle-point formulation, finite element method, augmented Lagrangians

1. Introduction

It is well known that the classical beam theory is based on the Euler-Bernoulli hypothesis. It states that plane sections perpendicular to the longitudinal axis of the beam before deformation remain plane, undeformed and perpendicular to the axis after deformation. The standard mathematical model for large deflection can be derived using the displacement field

$$u_x(x,y) = u(x) - y\theta(x), \quad u_y(x,y) = w(x), \quad u_z(x,y) = 0, \tag{1}$$

where u_x and u_y are axial and transverse displacement components of an arbitrary beam material point, w and u denotes transverse and horizontal displacements of the middle axis $y = 0$. θ is the bending angle and it holds $\theta = \tan^{-1}(w') \approx w'$. The motion in the z direction is of no interest. Under the assumption concerning the stress components $\sigma_x \neq 0, \sigma_y = 0$ one can derive (for details see e.g. [11, 13]) the following governing equations

$$\left(EA\left[u' + \frac{1}{2}(w')^2\right]\right)' = \tilde{f}, \tag{2}$$

$$(EI\,w'')'' - \left(EA\,w'\left[u' + \frac{1}{2}(w')^2\right]\right)' = \tilde{q}, \tag{3}$$

where E is the Young's modulus, A is the cross-section area, I is the moment of inertia, $\tilde{f}(x)$ is the distributed axial load (per unit length) and $\tilde{q}(x)$ is the distributed transverse load (per unit

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length). We can consider (2)–(3) as the 1D von Kármán equations. $\tilde{f} \equiv 0$ is a common case and it implies that after some rearrangements we obtain

$$(EI\,w'')'' - \left(EA\left[u' + \frac{1}{2}(w')^2\right]\right) w'' = \tilde{q}, \tag{4}$$

where the coefficient by w'' has a constant value and consequently (4) is only a linear equation.

This inadequacy was revised by D. Y. Gao in his paper [3] by the change of the assumption about the stress components to $\sigma_x \neq 0, \sigma_y \neq 0$. After a short recapitulation of the Gao's model we want to propose a suitable finite element (hereafter we will use abbr. FE) solution for this beam because the paper [3] has no remark about it. Then we are going to concern ourself with the system of this nonlinear beam plus unilateral Winkler foundation. First we have to establish a suitable formulation for the bending problem. Next we want to analyze the system in order to solve such problem and finally we intend to obtain a computational model for the considered problem.

As for the aforementioned foundation, the classical works were concerned with beams in fixed connection with a foundation. Such problems are *linear* provided the beam model is linear too. However, some applications have the different matter because the beam is not firmly connected with the given foundation. These are *nonlinear* problems regardless using beam model and in such cases we can speak about the *unilateral* foundation. Some works on this field have been done for the classical Euler-Bernoulli beam model, see e.g. [6, 8] and [12], but there are no papers concerning nonlinear beams with unilateral foundation.

2. The nonlinear beam by D. Y. Gao

Here we want to present only a brief introduction of the nonlinear beam model from the paper [3]. Let us consider an elastic beam whose cross section in the x–y plane is a rectangle $[0, \mathrm{L}] \times [-\mathrm{h}, \mathrm{h}]$ and in the y–z plane a rectangle $[-\mathrm{h}, \mathrm{h}] \times [0, \mathrm{b}]$, i.e. the beam's length is L, its thickness $2\mathrm{h}$ and its width b.

Displacements of such a beam are described by means of components (1). The Green-St Venant strain tensor for $x_1 = x$, $x_2 = y$, $x_3 = z$ has following components

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} u' - y\theta' + \frac{1}{2}(u' - y\theta')^2 + \frac{1}{2}(w')^2 & \frac{1}{2}(w' - \theta) - \frac{1}{2}(u' - y\theta')\theta \\ \frac{1}{2}(w' - \theta) - \frac{1}{2}(u' - y\theta')\theta & \frac{1}{2}\theta^2 \end{pmatrix}. \tag{5}$$

This gives us after neglecting small terms $(u' - y\theta')^2$, $(u' - y\theta')\theta$ and substituting $\theta = w'$

$$\varepsilon_{11} \equiv \epsilon_x = u' - yw'' + \frac{1}{2}(w')^2, \tag{6}$$

$$\varepsilon_{22} \equiv \epsilon_y = \frac{1}{2}(w')^2, \tag{7}$$

$$\varepsilon_{12} = 0. \tag{8}$$

More details can be found in [3]. The nonzero stress components now can be obtained by the following constitutive relation

$$\begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} \tag{9}$$

with ν denoting the Poisson's ratio.

Next we will suppose the beam is subject of a transversal load $\tilde{q}(x)$. The potential energy of a beam represented by a domain Ω is then as follows (see e.g. [13])

$$\Pi(u,w) = \frac{1}{2}\int_\Omega (\sigma_x\epsilon_x + \sigma_y\epsilon_y)\,\mathrm{d}\Omega - \int_0^{\mathrm{L}} \tilde{q}w\,\mathrm{d}x = \tag{10}$$

$$\frac{E}{2(1-\nu^2)}\int_\Omega (\epsilon_x^2(x,y) + 2\nu\epsilon_x(x,y)\epsilon_y(x) + \epsilon_y^2(x))\,\mathrm{d}\Omega - \int_0^{\mathrm{L}} \tilde{q}w\,\mathrm{d}x. \tag{11}$$

Using the Gâteaux derivatives (or first variations technique) for this functional we can get after some computation (see [3]) the system of two nonlinear equations for $x \in (0,\mathrm{L})$

$$u'' + (1+\nu)w'w'' = 0, \tag{12}$$

$$EI\,w^{IV} - 2hbE\,[(1+\nu)(2(w')^2 + u')w'' + \nu w'u''] = f, \tag{13}$$

assuming E is a constant, $I = \frac{2}{3}\mathrm{h}^3\mathrm{b}$ and denoting $f = (1-\nu^2)\tilde{q}$. The system can be reduced by integrating its first equation (12). We obtain

$$u' = -\frac{1}{2}(1+\nu)(w')^2 \tag{14}$$

and substituting this result into (13) we finally get

$$EI\,w^{IV} - E\alpha\,(w')^2 w'' = f \qquad \forall x \in (0,\mathrm{L}), \tag{15}$$

where

$$\alpha = 3\mathrm{hb}(1-\nu^2) \tag{16}$$

is a positive constant. The beam model described by the equation (15) is known as the *Gao beam* and it can be extended into a time-dependent model (see e.g. [4]).

3. Finite element model for the Gao beam

As the paper [3] contains only the beam theory, we are going to present here the FE approximation of the Gao beam. First we need a variational formulation of our problem. Let V be the space of kinematically admissible deflections v such that

$$H_0^2((0,\mathrm{L})) \subseteq V \subseteq H^2((0,\mathrm{L})). \tag{17}$$

Let us remember that the *Sobolev space* $H^2((0,\mathrm{L}))$ consists of those functions $v \in L^2((0,\mathrm{L}))$ for which derivatives v' and v'' (in the distribution sense) belong to the space $L^2((0,\mathrm{L}))$. The *Lebesgue space* $L^2((0,\mathrm{L}))$ is defined as the space of all measurable functions on $(0,\mathrm{L})$ which squares have a finite Lebesgue integral. And

$$H_0^2((0,\mathrm{L})) = \{v \in H^2((0,\mathrm{L}))\colon v(0) = v'(0) = 0 = v(\mathrm{L}) = v'(\mathrm{L})\} \tag{18}$$

(more information can be found e.g. in [1]). It is well known that the finite element method distinguishes between *natural* and *essential boundary conditions*. The first ones are contained in the space V, the second ones are built into the variational formulation. Without a loss of generality we will assume for definiteness the clamped boundary conditions, i.e. $V = H_0^2((0,\mathrm{L}))$, since another boundary conditions will not change in principle our approach.

From (15) after using integration by parts we can now immediately deduce

$$EI \int_0^{\mathrm{L}} w'' v'' \,\mathrm{d}x + \frac{1}{3} E\alpha \int_0^{\mathrm{L}} (w')^3 v' \,\mathrm{d}x = \int_0^{\mathrm{L}} f v \,\mathrm{d}x \qquad \forall v \in V. \tag{19}$$

This is in fact the equation for a stationary point of the potential energy of the Gao beam, which can be formally written as

$$\Pi_B'(w; v) = 0 \qquad \forall v \in V. \tag{20}$$

$\Pi_B'(w; v)$ denotes the Gâteaux derivative of Π_B at the point w in the direction v (see e.g. [1]). (19) with (20) imply that the functional of potential energy has the form

$$\Pi_B(w) = \frac{1}{2} EI \int_0^{\mathrm{L}} (w'')^2 \,\mathrm{d}x + \frac{1}{12} E\alpha \int_0^{\mathrm{L}} (w')^4 \,\mathrm{d}x - \int_0^{\mathrm{L}} f w \,\mathrm{d}x. \tag{21}$$

It is easy to prove that this functional is strictly convex. Then the equation (20) can be consequently rewritten as

$$\Pi_B(w) = \min_{v \in V} \Pi_B(v). \tag{22}$$

The problem of finding a function $w \in V$ such that (22) holds we will call the *variational formulation* of the Gao beam bending. The convexity implies the unique solution of the minimization problem (22) and also the fact that (22) can be equivalently represented by the equation (19).

Now we proceed to a FE discretization of our problem. For this purpose we have to construct some dividing of the interval $[0, \mathrm{L}]$ into subintervals $K_i = [x_{i-1}, x_i]$, where we have generated *nodes* $0 = x_0 < x_1 < \ldots < x_n = \mathrm{L}$. Formally, the *discrete problem* reads as follows:

Find $w_h \in V_h$ such that

$$EI \int_0^{\mathrm{L}} w_h'' v_h'' \,\mathrm{d}x + \frac{1}{3} E\alpha \int_0^{\mathrm{L}} (w_h')^3 v_h' \,\mathrm{d}x = \int_0^{\mathrm{L}} f v_h \,\mathrm{d}x \qquad \forall v_h \in V_h. \tag{23}$$

V_h is a finite-dimensional subspace of the given space V. In our case it has the form

$$V_h = \{v_h \in V : v_h|_{K_i} \in P_3(K_i) \quad \forall i = 1, \ldots, n\} \tag{24}$$

and contains piecewise polynomial functions from $C^1([0, \mathrm{L}])$, i.e. continuous on $[0, \mathrm{L}]$ together with its first derivatives. $P_3(K_i)$ denotes the set of cubic polynomials defined on K_i.

Now we can continue as it is usual for the standard FE beam model. We define the *Hermite basis functions* for our space (24) (see e.g. [8]) and afterwards the *shape functions* on a single element, which are beneficial from the practical computation point of view (see e.g. [8, 10]). But contrary to the standard FE solution process the second term in (23) prevents us to obtain a system of linear equations, as it is a rule in the classical beam theory. In matrix form we get formally

$$[\boldsymbol{K}_1 + \boldsymbol{K}_2(\boldsymbol{w})]\, \boldsymbol{w} = \boldsymbol{f}. \tag{25}$$

Into the vector $\boldsymbol{w}$ we assembled all the unknowns. This system contains the matrix $\boldsymbol{K}_1$ from the first integral in (23), which is well known from the linear FE model, and the matrix $\boldsymbol{K}_2$ from the second integral in (23), which depends on the vector of unknowns $\boldsymbol{w}$ and therefore cannot be evaluated explicitly (similar cases are described e.g. in [11]). Formulas concerning this matrix are quite cumbersome and we omit them here. Traditional method for solution of (25) is the Newton method (see e.g. [9, 11]). Of course, the infamous property of the Newton method is its sensitivity to a good initial guess, which can occasionally cause divergence of our computational process.

A fair alternative is return to the minimization problem (22) instead of the nonlinear system solution. First we formulate the discrete optimization problem to (22) as follows:

Find $w_h \in V_h$ such that

$$\Pi_B(w_h) = \min_{v_h \in V_h} \Pi_B(v_h). \tag{26}$$

The same discretization process as above leads here not to a system of equations but to the minimization of the strictly convex function of N unknowns

$$F_B(\boldsymbol{w}) = \min_{\boldsymbol{v} \in \mathbb{R}^N} F_B(\boldsymbol{v}), \tag{27}$$

which gradient is formally done by the expression

$$\nabla F_B(\boldsymbol{v}) = [\boldsymbol{K}_1 + \boldsymbol{K}_2(\boldsymbol{v})]\,\boldsymbol{v} - \boldsymbol{f}. \tag{28}$$

The methods such as the conjugate gradient method or the BFGS method require only computation of the gradient and some inexact line-search algorithm to determine a step size. For the details we encourage the gentle reader to look through some book concerning optimization methods, e.g. [9].

4. Problem with an unilateral foundation

In this section we are going to present a new extension of Gao's work. We will deal with bending of the Gao beam resting on the Winkler foundation. The classical Winkler model is based on the assumption of a linear force-deflection relationship and a fixed connection between the beam and the foundation. Let k_F is the foundation modulus, which will be supposed constant. Then, with respect to (15), the requested equation is

$$EI\, w^{IV} - E\alpha\,(w')^2 w'' + k_F w = f \qquad \forall x \in (0, \mathrm{L}). \tag{29}$$

Very easy is obtaining the variational formulation, because the potential energy of the Winkler foundation is

$$\Pi_F(v) = \frac{1}{2} k_F \int_0^{\mathrm{L}} v^2\,\mathrm{d}x \tag{30}$$

and regarding (21) consequently for the total energy holds

$$\Pi_{B+F}(v) = \frac{1}{2} EI \int_0^{\mathrm{L}} (v'')^2\,\mathrm{d}x + \frac{1}{12} E\alpha \int_0^{\mathrm{L}} (v')^4\,\mathrm{d}x - \int_0^{\mathrm{L}} f v\,\mathrm{d}x + \frac{1}{2} k_F \int_0^{\mathrm{L}} v^2\,\mathrm{d}x. \tag{31}$$

The variational formulation afterwards reads as follows:

Find $w \in V$ such that

$$\Pi_{B+F}(w) = \min_{v \in V} \Pi_{B+F}(v) \tag{32}$$

and, since the strict convexity still holds, this can be equivalently expressed as

$$EI \int_0^{\mathrm{L}} w'' v''\,\mathrm{d}x + \frac{1}{3} E\alpha \int_0^{\mathrm{L}} (w')^3 v'\,\mathrm{d}x + k_F \int_0^{\mathrm{L}} w v\,\mathrm{d}x = \int_0^{\mathrm{L}} f v\,\mathrm{d}x \qquad \forall v \in V. \tag{33}$$

Next our attention will be focused on the so-called *unilateral case*, when the foundation and the beam are not interconnected. This case was studied e.g. in [6] and [12] for the linear

Euler-Bernoulli beam model. We will assume that the vertical axis is turned down. Applying then the technique from the mentioned works, we can rewrite (33) as follows

$$EI \int_0^{\mathrm{L}} w'' v'' \, \mathrm{d}x + \frac{1}{3} E\alpha \int_0^{\mathrm{L}} (w')^3 v' \, \mathrm{d}x + k_F \int_0^{\mathrm{L}} w^+ v \, \mathrm{d}x = \int_0^{\mathrm{L}} f v \, \mathrm{d}x \qquad \forall v \in V, \tag{34}$$

where $w^+(x) = \frac{1}{2}(w(x) + |w(x)|) = \max\{0, w(x)\}$. Of course, we are able to write the variational formulation for the unilateral problem in the form:

Find $w \in V$ such that

$$\widetilde{\Pi}_{B+F}(w) = \min_{v \in V} \widetilde{\Pi}_{B+F}(v), \tag{35}$$

where

$$\widetilde{\Pi}_{B+F}(v) = \frac{1}{2} EI \int_0^{\mathrm{L}} (v'')^2 \, \mathrm{d}x + \frac{1}{12} E\alpha \int_0^{\mathrm{L}} (v')^4 \, \mathrm{d}x + \frac{1}{2} k_F \int_0^{\mathrm{L}} (v^+)^2 \, \mathrm{d}x - \int_0^{\mathrm{L}} f v \, \mathrm{d}x. \tag{36}$$

But from now we are going to follow a different way compared to the cited papers.

Let us define a *problem decomposition* using a linear relationship, which in general has the form

$$Bv = q \qquad v \in V, q \in Q. \tag{37}$$

B is a linear continuous operator from V into Q. The decomposition naturally split our problem into two pieces: the beam and the foundation. For our case we choose $Q = L^2((0, \mathrm{L}))$ and B as the identity, more precisely the canonical mapping from V into Q. Thereby we get a new variable q joined with the foundation and defined by

$$v = q \qquad v \in V, \ q \in Q, \tag{38}$$

while the beam will be described by the old variable v. After that we have the new functional

$$\widehat{\Pi}_{B+F}(v, q) = \frac{1}{2} EI \int_0^{\mathrm{L}} (v'')^2 \, \mathrm{d}x + \frac{1}{12} E\alpha \int_0^{\mathrm{L}} (v')^4 \, \mathrm{d}x + \frac{1}{2} k_F \int_0^{\mathrm{L}} (q^+)^2 \, \mathrm{d}x - \int_0^{\mathrm{L}} f v \, \mathrm{d}x \tag{39}$$

defined on the set

$$W = \{\{v, q\} \in V \times Q \colon v = q\} \tag{40}$$

and the variational formulation of the problem with unilateral foundation is then as follows:

Find $\{w, p\} \in W$ such that

$$\widehat{\Pi}_{B+F}(w, p) = \min_{\{v,q\} \in W} \widehat{\Pi}_{B+F}(v, q). \tag{41}$$

It is evident that (41) is equivalent to (35). This way we follow the main idea from [7] and this is a part of a more general strategy called the *decomposition-coordination method* from [5].

But there is some inconvenience in (41). The new formulation represents a *constrained optimization* problem. To handle it right, we must define the *Lagrangian* for our problem by

$$\mathcal{L}(v, q, \mu) = \widehat{\Pi}_{B+F}(v, q) + \int_0^{\mathrm{L}} \mu (v - q) \, \mathrm{d}x \qquad v \in V, \ q \in Q, \ \mu \in \Lambda, \tag{42}$$

where μ is the *Lagrange multiplier* associated with the constraint $v = q$ and $\Lambda = L^2((0, \mathrm{L}))$. It can be proved (see e.g. [2]), that our problem (41) can be reformulated as the so-called *saddle-point problem* for the Lagrangian (42):

Find $\{w, p, \lambda\} \in V \times Q \times \Lambda$ such that

$$\mathcal{L}(w, p, \mu) \leq \mathcal{L}(w, p, \lambda) \leq \mathcal{L}(v, q, \lambda) \quad \forall v \in V,\ q \in Q,\ \mu \in \Lambda. \tag{43}$$

In our case (43) can be equivalently expressed as follows

$$\mathcal{L}(w, p, \lambda) = \inf_{\{v,q\} \in V \times Q} \sup_{\mu \in \Lambda} \mathcal{L}(v, q, \mu) = \sup_{\mu \in \Lambda} \inf_{\{v,q\} \in V \times Q} \mathcal{L}(v, q, \mu). \tag{44}$$

From here we can observe, that it is possible to obtain the unknowns w, p by some minimization. Therefore, by this way we transformed the constrained problem (41) into an *unconstrained* one at the cost of the additional unknown, i.e. the Lagrange multiplier λ.

By means of Gâteaux derivatives (or first variations) of the Lagrangian $\mathcal{L}$ with respect to q and μ we obtain at the point $\{w, p, \lambda\}$ the following results

$$w = p, \quad \lambda = k_F\, p^+ \qquad \text{a.e. in } L^2((0, \mathrm{L})). \tag{45}$$

The first one was expected regarding (38), the second one gives us the interpretation of the Lagrange multiplier λ.

Finally, we must mention the question of the existence of a saddle point $\{w, p, \lambda\}$. In *infinite dimensions* this question coincide with the question of the existence of a Lagrange multiplier λ and it is, however, somewhat problematical. Sufficient conditions to assure the existence of the multiplier λ would be found e.g. in [2]. The problem considered in this article fulfills these conditions and (44) has therefore a solution.

5. Solution of the given problem

Now we consider some possibilities how to solve our problem (44). There are two principal ways to this objective. The first one consists in transformation our problem into the system of nonlinear equations. The second way is based on using of optimization methods to find a saddle point of (42). We can recognize the situation is in a certain manner similar to that we encountered by finding solution for bending of the Gao beam.

The first way uses a transformation to a mixed complementarity problem and will be omitted in this article as it would be rather extensive (for Euler-Bernoulli beam this approach was realized e.g. in [7]). So that we will concern our attention to the second possibility for solution of our problem (44) which consists in taking advantage of optimization methods. Despite the fact that the saddle-point problem is not a true optimization problem, we have a good opportunity in combining two methods. The first one is the *Uzawa algorithm* for finding saddle points and the second one is the so-called *augmented Lagrangian method*. Hereafter we will mainly follow [5].

The *augmented Lagrangian* $\mathcal{L}_r$ is defined in our case for any $r > 0$ by

$$\mathcal{L}_r(v, q, \mu) = \mathcal{L}(v, q, \mu) + \frac{r}{2} \int_0^{\mathrm{L}} (v - q)^2 \,\mathrm{d}x \tag{46}$$

with $\mathcal{L}$ given by (42). Next we can introduce the saddle-point problem for this augmented Lagrangian:

Find $\{w, p, \lambda\} \in V \times Q \times \Lambda$, with V from (17) and $Q, \Lambda = L^2((0, \mathrm{L}))$, such that

$$\mathcal{L}_r(w, p, \mu) \leq \mathcal{L}_r(w, p, \lambda) \leq \mathcal{L}_r(v, q, \lambda) \qquad \forall v \in V,\ q \in Q,\ \mu \in \Lambda. \tag{47}$$

An advantageousness of the augmented Lagrangian method is given by the fact that we can state the following result (for the proof see [5]):

Suppose $\{w, p, \lambda\}$ is a saddle point of $\mathcal{L}$ on $V \times Q \times \Lambda$. Then $\{w, p, \lambda\}$ is a saddle point of $\mathcal{L}_r$ for every $r > 0$, and vice versa. Furthermore w is a solution of the original problem (35), (36) and we have $p = w$.

Hence we can interchange the problems (43) and (47) and from the computational point of view the second one will be much more convenient. We have to notice, that in *finite dimensions* the existence of a saddle point is assured, since we minimize under a linear equality constraint.

Considering all things, to solve the problem (35), (36) we need to determine the saddle points of the Lagrangian $\mathcal{L}$ from (42) and consequently the saddle points of the augmented Lagrangian $\mathcal{L}_r$ from (46). This can be attained with the help of a variant of the Uzawa algorithm. The rather complicated problem in the implementation of such an algorithm presents the solution of the minimization problem for $\mathcal{L}_r$ with respect to $\{v, q\}$ at each iteration. A frequently used solution procedure consists of using the block relaxation method which leads to the following algorithm

$p^0 \in Q, \lambda^1 \in \Lambda$ are given,
then for $n = 1, 2, \ldots$
determine w^n, p^n as follows:
find $w^n \in V$ such that
$\mathcal{L}_r(w^n, p^{n-1}, \lambda^n) \le \mathcal{L}_r(v, p^{n-1}, \lambda^n) \qquad \forall v \in V,$
find $p^n \in Q$ such that
$\mathcal{L}_r(w^n, p^n, \lambda^n) \le \mathcal{L}_r(w^n, q, \lambda^n) \qquad \forall q \in Q,$
determine λ^{n+1} as follows:
$\lambda^{n+1} = \lambda^n + \rho(w^n - p^n) \qquad \rho > 0.$

Under quite general assumptions we have convergence of this algorithm under the condition $0 < \rho < ((1 + \sqrt{5})/2)r$. The proof may be found in [5]. Let us remark that for our functional (36) aforementioned assumptions are fulfilled. The good choice for ρ seems to be in most cases $\rho = r$. Moreover, then we are able to implement some modification into our algorithm. From the equation for the minimization of $\mathcal{L}_r$ with respect to q we get

$$r(p^n - w^n) = \lambda^n - k_F(p^n)^+ \qquad \text{a.e. in} L^2((0, \mathrm{L})). \tag{48}$$

This result helps us to adjust the Uzawa step as follows

$$\lambda^{n+1} = \lambda^n + r(w^n - p^n) = \lambda^n + k_F(p^n)^+ - \lambda^n = k_F(p^n)^+ \tag{49}$$

and the last row of our algorithm can now be rewritten according to (49).

Finally, for computational purposes we must define suitable approximations of the infinite-dimensional spaces V, Q and Λ. Let us denote their finite-dimensional subspaces as V_h, Q_h and Λ_h. V_h will be the same as in (24), Q_h and Λ_h can be chosen as

$$Q_h = \Lambda_h = \{q_h \in L^2((0, \mathrm{L}))\colon q_h|_{K_i} \in P_0(K_i) \qquad \forall i = 1, \ldots, n\}, \tag{50}$$

i.e. these spaces consist of piecewise constant functions.

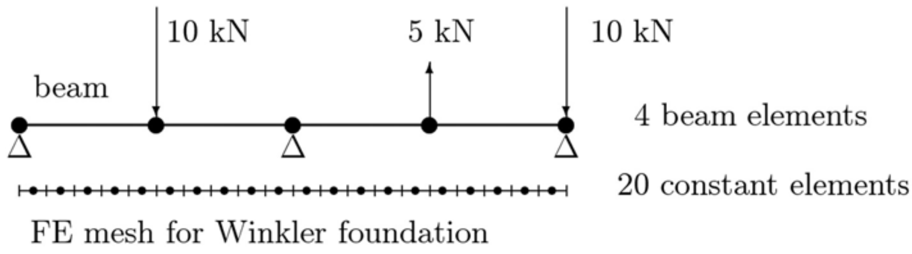

Fig. 1. Sketch for the example

6. Example

Here we want to illustrate the above explained theory and methods on a simple example. Let us consider a beam of the length $\mathrm{L} = 4$ m with three supports at $x = 0$ m, $x = 2$ m and $x = 4$ m and resting on a Winkler foundation. Data for the beam and the foundation are given as follows: $EI = 2 \times 10^7\ \mathrm{N} \cdot \mathrm{m}^2$, $\mathrm{h} = 0.25$ m, $\mathrm{b} = 0.4$ m, $\nu = 0.3$, $k_F = 2 \times 10^7\ \mathrm{N} \cdot \mathrm{m}^{-2}$. Three isolated forces are acting at $x = 1$ m, $x = 3$ m and $x = 4$ m as it can be seen from Fig. 1, where is also an example how finite element meshes are constructed (dots denote element nodes).

Table 1. Results for the example

number of elements		EB linear beam				Gao beam			
		classical WF		unilateral WF		classical WF		unilateral WF	
beam	found.	u max	u min	u max	u min	u max	u min	u max	u min
4	40	6.237	−4.525	6.433	−5.036	5.639	−4.093	5.805	−4.544
4	100	6.236	−4.524	6.431	−5.035	5.637	−4.091	5.803	−4.542
4	400	6.236	−4.524	6.431	−5.035	5.636	−4.090	5.802	−4.541
8	40	6.238	−4.526	6.433	−5.036	5.639	−4.092	5.805	−4.544
8	104	6.237	−4.525	6.432	−5.036	5.637	−4.091	5.803	−4.542
8	400	6.237	−4.525	6.432	−5.035	5.637	−4.091	5.803	−4.542

Results for extreme displacement values are given in the Table 1, presented numbers should be multiplied by the scaling factor 10^{-5} m. The table contains results for the Euler-Bernoulli (abbr. EB) beam and for the Gao beam, both with the classical Winkler foundation (abbr. WF) and unilateral foundation. Different meshes give the quite similar numbers and we can observe something like convergence of the numerical values. The nonlinear beam proves itself as more stiff, which we could expect e.g. from (25) due to an additional stiffness matrix $\boldsymbol{K}_2$.

7. Conclusion

We proposed here the new way how to formulate and solve the problem of bending of the nonlinear Gao beam while the beam is resting on the unilateral Winkler foundation, which is not connected with the beam. The beam and the foundation have their own finite elements and element meshes which are closer to their physical fundamentals as it is in contact problems. But we are not forced to solve a contact problem. Our solution uses a saddle-point formulation and represents some compromise between a contact solution technique and a standard practice. It can be realized either through the application of methods for a system of nonlinear nondifferentiable equations, namely the nonsmooth Newton method, or by the help of the augmented Lagrangian method. The numerical example demonstrated some possibilities of the new solution method.

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