

Well-posedness and stabilization of a nonuniform wind turbine tower with distributed delay

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Abstract

This study focuses on the boundary control of a wind turbine tower considering distributed delay, modeled as a non-uniform Euler-Bernoulli beam. A control input is imposed at the top boundary to suppress tower vibrations. The well-posedness result is achieved through the semi-group approach, and employing Lyapunov method confirms the stabilization of the tower to a small neighborhood of zero. Numerical simulations are provided to visually depict and validate the efficacy of the proposed control.

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1. Introduction

This paper focuses on studying the vibration control in a wind turbine tower subjected to unpredictable wind loads. The tower characterization includes a non-uniform Euler-Bernoulli beam model with distributed delay, using Hamilton's principle [24], which leads to the following problem:

$$\begin{aligned}
 \rho A(r)y_{tt}(r,t) + (EI(r)y_{rr}(r,t))_{rr} - (T(r)y_r(r,t))_r + \eta_a y_t(r,t) \\
 + \int_{\tau_1}^{\tau_2} \eta_b(s)y_t(r,t-s)ds = f(r,t), \quad r \in (0,L), t > 0, \\
 y(0,t) = y_r(0,t) = y_{rr}(L,t) = 0, \\
 My_{tt}(L,t) - (EI(r)y_{rr})_r(L,t) + T(L)y_r(L,t) = U(t) + g(t), \quad t > 0, \\
 y(r,0) = y^1(r), \quad y_t(r,0) = y^2(r), \quad y_t(r,-t) = y^3(r,t), \quad r \in (0,L), t \in (0,\tau_2),
 \end{aligned} \tag{1}$$

where $y(r,t)$ and $f(r,t)$ represent the transverse tower displacements and distributed loads along the tower. The subscripts denote partial derivatives. The parameters ρ , L , E , η_a , and M correspond to the mass density, length, Young modulus, damping coefficient, and point mass concentrated at the tower's upper boundary, respectively. The integral term represents the distributed delay, where τ_1 , τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. The functions $I(r)$, $T(r)$, and $A(r)$ represent the nonuniform moment, nonuniform axial tension, and nonuniform area of the tower's cross-section. Additionally, $\eta_b(t)$, $U(t)$, and $g(t)$ stand for the weight of the delay, boundary control, and disturbance at $r = L$.

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More specifically, $f(r, t)$ is a distributed transverse load acting continuously along the beam over the spatial domain $r \in (0, L)$. In contrast, $g(t)$ represents a concentrated transverse force applied at the free end $r = L$. Thus, $f(r, t)$ acts throughout the interior of the beam and appears as a forcing term in the governing differential equation, whereas $g(t)$ acts only at the boundary and consequently enters the model through the corresponding boundary condition.

The function $T(r)$ represents the axial tension in the beam and is allowed to vary with the spatial coordinate r . Such a nonuniform axial tension arises when the axial force changes along the beam, for instance as a result of spatially varying prestress, loading conditions, or geometric and material properties. Its contribution to the transverse dynamics is represented by the term $-(T(r)y_r(r, t))_r$. Accordingly, this term accounts for the effect of the spatial variation of the axial tension on the transverse motion of the beam.

The boundary control of slender structures, particularly non-uniform Euler-Bernoulli beams, has attracted considerable attention in recent years owing to their distinctive geometric properties and wide range of engineering applications. The primary objective of controlling these flexible structures is to suppress or reduce vibrations arising from non-uniform material properties or external disturbances. Related studies have addressed boundary control in wind turbine towers [23], Euler-Bernoulli beams [7, 8, 12] and Timoshenko beams [6].

In many studies, internal dissipation mechanisms are employed to achieve system stability, including viscoelastic damping [2, 15, 21] and infinite memory effects [3–5]. When such internal dissipation mechanisms are insufficient to provide the desired energy decay rate, boundary control can be employed as an effective stabilization mechanism. This approach has been investigated for wave equations [16, 35], one-dimensional beam models [27, 46], two-dimensional beam models [31, 32] and related stability problems involving adaptive control [29, 36]. In addition, experimental investigations providing evidence of structural vibrations have been reported in [10, 13, 37].

Previous research has shown that delay terms appearing in the first equation of (1) may act as destabilizing perturbations and can adversely affect the stability of the system, see [41]. To mitigate the destabilizing effects of delay, additional viscous dissipation has been introduced in several studies. In particular, viscous damping has been considered in systems with general delay terms [19, 20, 44], nonlinear delay terms [34], nonlocal delay terms [43] and fractional delay terms [9, 14]. Related stabilization results for systems involving delay and associated dissipative mechanisms can also be found in [25, 26]. Several studies have investigated the effect of time delays in boundary and internal feedback on the stabilization of various systems; see, for example, [17, 18, 39].

The system (1), without delay and tension, has been previously studied in [24] with control

$$U = -\frac{k}{\beta} [\alpha L y_r + \beta y_t],$$

where $\alpha, \beta > 0$ and $k > 0$ represent the control gain. Furthermore, the same problem, but with a damping coefficient function $\eta_\alpha(r)$, was investigated in [45] with control

$$U = -\alpha y + \beta y_t.$$

In both cases, an exponential stabilization result was established. A similar result was obtained in [1] and [22] by neglecting internal damping and relying on boundary control only. For related works dealing with constant coefficients, we refer the reader to [28].

To the best of our knowledge, the stabilization of non-uniform structures with distributed delay, as described by system (1), has not been previously investigated. The main mathematical

difficulty arises from the simultaneous presence of variable spatial coefficients and delay terms. In particular, the nonconstant coefficients generate additional terms in the energy estimates and lead to variable-coefficient contributions that cannot be handled directly by the standard estimates available for constant-coefficient systems. These effects complicate the construction of a coercive energy functional and the derivation of a suitable dissipation estimate. To overcome these difficulties, we introduce an appropriate weighted Lyapunov functional and combine it with carefully designed estimates for the variable-coefficient and delay terms. This approach enables us to establish the required energy decay estimate and, consequently, the stabilization of the considered system.

The remainder of the paper is organized as follows: Section 2 introduces the fundamental assumptions and auxiliary lemmas required for the analysis. Section 3 is devoted to establishing the well-posedness of the problem. In Section 4, we prove the stabilization result, while Section 5 presents numerical simulations that illustrate and corroborate the theoretical findings.

2. Preliminary

Here, we introduce the assumptions and preliminary results needed to state and prove our main results:

(H1) Regarding the disturbance loads, we assume that $f \in L^\infty(\mathbb{R}_+; L^2(0, L))$ and $g \in L^\infty(\mathbb{R}_+)$, that is, there exist positive constants $\bar{f}$ and $\bar{g}$ such that

$$\int_0^L (f(r, t))^2 dr \leq \bar{f} \quad \text{and} \quad |g(t)|^2 \leq \bar{g}, \quad t \geq 0.$$

(H2) For the parameters A , T , and I , we assume that there exist positive constants $A_{\min}$, $A_{\max}$, $T_{\min}$, and $T_{\max}$ such that

$$0 < A_{\min} \leq A(r) \leq A_{\max}, \quad 0 < T_{\min} \leq T(r) \leq T_{\max}, \quad 0 < I(r), \quad \forall r \in (0, L).$$

(H3) Let $\eta_a > 0$. For the delay weight function [30, 33], we assume that $\eta_b \in L^\infty(\tau_1, \tau_2; \mathbb{R}_+)$ satisfying

$$\eta_a > \int_{\tau_1}^{\tau_2} \eta_b(s) ds.$$

Then, there exists a constant $\eta_c > 0$, sufficiently small, such that

$$\eta_a - \int_{\tau_1}^{\tau_2} \left(\eta_b(s) + \frac{\eta_c}{2} \right) ds \geq 0.$$

More precisely, η_c is chosen to satisfy

$$0 < \eta_c \leq \frac{2}{\tau_2 - \tau_1} \left(\eta_a - \int_{\tau_1}^{\tau_2} \eta_b(s) ds \right).$$

The energy of system (1) is

$$\begin{aligned} e(t) = & \frac{M}{2} (y_t(L, t))^2 + \frac{\rho}{2} \int_0^L A(r) (y_t(r, t))^2 \, dr \\ & + \frac{E}{2} \int_0^L I(r) (y_{rr}(r, t))^2 \, dr + \frac{1}{2} \int_0^L T(r) (y_r(r, t))^2 \, dr. \end{aligned} \quad (2)$$

This represents the classic energy expression, where the first line in (2) expresses kinetic energy, while the second line is for potential energy. For control U , we propose

$$U(t) = -ky_t(L, t), \quad k \in \mathbb{R}_+. \quad (3)$$

Lemma 1. The energy (2) satisfies

$$\begin{aligned} e'(t) = & -\eta_a \int_0^L (y_t(r, t))^2 \, dr - \int_0^L y_t(r, t) \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t-s) \, ds \, dr \\ & + (U(t) + g(t)) y_t(L, t) + \int_0^L y_t f(r, t) \, dr, \end{aligned} \quad (4)$$

where the prime $(\cdot)'$ denotes differentiation with respect to time t .

Proof. Differentiating the energy functional (2) with respect to time gives

$$\begin{aligned} e'(t) = & M y_t(L, t) y_{tt}(L, t) + \rho \int_0^L A(r) y_t(r, t) y_{tt}(r, t) \, dr \\ & + \int_0^L [EI(r) y_{rr}(r, t) y_{rrt}(r, t) + T(r) y_r(r, t) y_{rt}(r, t)] \, dr. \end{aligned}$$

Integrating by parts and employing the boundary conditions in (1), we obtain

$$\begin{aligned} \int_0^L EI(r) y_{rr} y_{rrt} \, dr &= - \int_0^L (EI(r) y_{rr})_r y_{rt} \, dr \\ &= - (EI(L) y_{rr})_r y_t(L, t) + \int_0^L (EI(r) y_{rr})_{rr} y_t(r, t) \, dr \end{aligned}$$

and

$$\int_0^L T(r) y_r y_{rt}(r, t) \, dr = T(L) y_r y_t(r, t) - \int_0^L (T(r) y_r)_r y_t(r, t) \, dr.$$

Consequently,

$$\begin{aligned} e'(t) = & \int_0^L \left[\rho A(r) y_{tt}(r, t) + (EI(r) y_{rr})_{rr}(r, t) - (T(r) y_r)_r(r, t) \right] y_t(r, t) \, dr \\ & + [My_{tt}(L, t) - (EI(L) y_{rr})_r(L, t) + T(L) y_r(L, t)] y_t(L, t). \end{aligned}$$

Finally, substituting the governing equations (1) into the above identity yields

$$\begin{aligned} e'(t) = & - \int_0^L \eta_a (y_t(r, t))^2 \, dr - \int_0^L y_t(r, t) \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t-s) \, ds \, dr \\ & + \int_0^L f(r, t) y_t(r, t) \, dr + (U(t) + g(t)) y_t(L, t). \quad \square \end{aligned}$$

In order to incorporate the distributed delay term into the energy of system (1), we introduce the modified energy as follows:

$$\begin{aligned} E(t) = & \frac{M}{2} (y_t(L, t))^2 + \frac{\rho}{2} \int_0^L A(r) (y_t(r, t))^2 \, dr + \frac{E}{2} \int_0^L I(r) (y_{rr}(r, t))^2 \, dr \\ & + \frac{1}{2} \int_0^L T(r) (y_r(r, t))^2 \, dr + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} s [\eta_b(s) + \eta_c] \int_0^1 (y_t(r, t-ps))^2 \, dp \, ds \, dr. \quad (5) \end{aligned}$$

Lemma 2. The energy E satisfies

$$\begin{aligned} E'(t) \leq & - \left[k - \frac{\delta}{2} \right] (y_t(L, t))^2 - \left[\frac{\eta_d}{A_{\max}} - \frac{\delta}{2A_{\min}} \right] \int_0^L A(r) (y_t(r, t))^2 \, dr \\ & - \frac{\eta_c}{2} \int_0^L \int_{\tau_1}^{\tau_2} (y_t(r, t-s))^2 \, ds \, dr + \frac{\bar{f} + \bar{g}}{2\delta}, \quad (6) \end{aligned}$$

where $\delta > 0$ and $\eta_d = \eta_a - \int_{\tau_1}^{\tau_2} [\eta_b(r) + \eta_c/2] \, dr$.

Proof. From Lemma 1, we have

$$\begin{aligned} E'(t) = & - \eta_a \int_0^L (y_t(r, t))^2 \, dr - \int_0^L y_t(r, t) \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t-s) \, ds \, dr \\ & + (U(t) + g(t)) y_t(L, t) + \int_0^L y_t(r, t) f(r, t) \, dr \\ & + \int_0^L \int_{\tau_1}^{\tau_2} s [\eta_b(s) + \eta_c] \int_0^1 y_t(r, t-ps) y_{tt}(r, t-ps) \, dp \, ds \, dr. \quad (7) \end{aligned}$$

Using Young's inequality, the first delay term in (7) can be estimated as

$$\begin{aligned} \int_0^L y_t(r, t) \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t-s) \, ds \, dr &\leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} \eta_b(s) \, ds \right) \int_0^L (y_t(r, t))^2 \, dr \\ &\quad + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} \eta_b(s) (y_t(r, t-s))^2 \, ds \, dr. \end{aligned} \quad (8)$$

Moreover, for any $\delta > 0$, Young's inequality and assumption (H1) give

$$g(t) y_t(L, t) \leq \frac{\delta}{2} (y_t(L, t))^2 + \frac{1}{2\delta} (g(t))^2 \leq \frac{\delta}{2} (y_t(L, t))^2 + \frac{\bar{g}}{2\delta}. \quad (9)$$

Similarly,

$$\begin{aligned} \int_0^L y_t(r, t) f(r, t) \, dr &\leq \frac{\delta}{2} \int_0^L (y_t(r, t))^2 \, dr + \frac{1}{2\delta} \int_0^L (f(r, t))^2 \, dr \\ &\leq \frac{\delta}{2A_{\min}} \int_0^L A(r) (y_t(r, t))^2 \, dr + \frac{\bar{f}}{2\delta}, \end{aligned} \quad (10)$$

where $A(r) \geq A_{\min} > 0$.

For the last term in (7), following [38], we introduce the auxiliary variable $z(r, p, t, s) = y_t(r, t - ps)$. By the chain rule, $y_{tt}(r, t - ps) = -z_p(r, p, t, s)/s$. Consequently,

$$\begin{aligned} \int_0^1 y_t(r, t - ps) y_{tt}(r, t - ps) \, dp &= -\frac{1}{s} \int_0^1 z(r, p, t, s) z_p(r, p, t, s) \, dp \\ &= -\frac{1}{2s} \left[(z(r, 1, t, s))^2 - (z(r, 0, t, s))^2 \right] \\ &= -\frac{1}{2s} \left[(y_t(r, t - s))^2 - (y_t(r, t))^2 \right]. \end{aligned} \quad (11)$$

Combining (7)–(11) and using the assumption (H2), we obtain

$$\begin{aligned} E'(t) &\leq - \left[\frac{\eta_d}{A_{\max}} - \frac{\delta}{2A_{\min}} \right] \int_0^L A(r) (y_t(r, t))^2 \, dr + \frac{\delta}{2} (y_t(L, t))^2 \\ &\quad - \frac{\eta_c}{2} \int_0^L \int_{\tau_1}^{\tau_2} (y_t(r, t-s))^2 \, ds \, dr + U(t) y_t(L, t) + \frac{\bar{f} + \bar{g}}{2\delta}, \end{aligned}$$

where $\eta_d = \eta_a - \int_{\tau_1}^{\tau_2} [\eta_b(s) + \frac{\eta_c}{2}] \, ds$.

Finally, substituting the proposed boundary control (3) into the above inequality yields (6). $\square$

3. Well posedness

At this point, following the approach in [38], we introduce the following change of variables:

$$z(r, p, t, s) = y_t(r, t - ps), \quad (0, L) \times (0, 1) \times \mathbb{R}_+ \times (\tau_1, \tau_2).$$

Subsequently, problem (1) is equivalent to

$$\begin{aligned} \rho A(r) y_{tt}(r, t) + (EI(r) y_{rr}(r, t))_{rr} - (T(r) y_r(r, t))_r + \eta_a y_t(r, t) \\ + \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t - s) ds = f(r, t), \quad r \in (0, L), t > 0, \\ z_t(r, p, t, s) = -s^{-1} z_p(r, p, t, s), \quad (0, L) \times (0, 1) \times \mathbb{R}_+ \times (\tau_1, \tau_2), \\ y(0, t) = y_r(0, t) = y_{rr}(L, t) = 0, \\ M y_{tt}(L, t) - (EI(r) y_{rr})_r(L, t) + T(L) y_r(L, t) = U(t) + g(t), \quad t > 0, \\ y(r, 0) = y^1(r), \quad y_t(r, 0) = y^2(r), \quad y_t(r, -t) = y^3(r, t), \quad r \in (0, L), t \in (0, \tau_2). \end{aligned} \quad (12)$$

For consistency and simplicity of notation in this section, the unknown functions will be written without explicitly displaying their spatial and temporal variables; that is, $y = y(r, t)$ is simply denoted by y . The variables are displayed explicitly whenever boundary traces are involved, for example, $y(L, t)$ or $y|_{r=L}$, etc.

To apply semigroup theory, we consider $V = \{y \in H^2(0, L), y(0, t) = y_r(0, t) = 0\}$ and

$$H = \left\{ (y, y_t, z, y(L, t), y_t(L, t))^T, y \in V, y_t \in L^2(0, L), y(L, t), y_t(L, t) \in \mathbb{R} \right\},$$

associated with the inner product

$$\begin{aligned} (v, \bar{v})_H = \frac{1}{2} \int_0^L (\rho A(r) y_t \bar{y}_t + EI(r) y_{rr} \bar{y}_{rr} + T(r) y_r \bar{y}_r) dr \\ + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} s [\eta_b(s) + \eta_c] \int_0^1 z \bar{z} dp ds dr + \frac{M}{2} y_t(L, t) \bar{y}_t(L, t) + \frac{1}{2} y(L, t) \bar{y}(L, t), \end{aligned}$$

where $v = (y, y_t, z, y(L, t), y_t(L, t))^T$, $\bar{v} = (\bar{y}, \bar{y}_t, \bar{z}, \bar{y}(L, t), \bar{y}_t(L, t))^T \in H$.

Therefore, the controlled system (12) can be reformulated in the abstract form

$$\begin{aligned} v'(t) &= Av(t) + F(t), \quad t > 0, \\ v(0) &\in H, \end{aligned} \quad (13)$$

where $F(t) = (0, f(r, t), 0, 0, g(t))^T$ represents the external disturbance terms. The operator A incorporates both the tower dynamics and the boundary feedback control (3), and is given by

$$Av = \begin{pmatrix} y_t \\ \frac{1}{\rho A(r)} \left[(T(r) y_r)_r - (EI(r) y_{rr})_{rr} - \eta_a y_t - \int_{\tau_1}^{\tau_2} \eta_b(s) z(r, 1, t, s) ds \right] \\ -s^{-1} z_p(r, p, t, s) \\ y_t(L, t) \\ \frac{1}{M} \left[(EI(r) y_{rr})_r|_{r=L} - T(L) y_r(L, t) - k y_t(L, t) \right] \end{pmatrix}$$

with domain $D(A) = \{(y, y_t, z, y(L, t), y_t(L, t)) \in V \cap H^4(0, L) \times V \times \mathbb{R}^2, y_{rr}(L, t) = 0\}$.

Definition 1. Let $v(0) \in H$. A function $v \in \mathcal{C}([0, \infty); H)$ is called a weak solution of problem (13) on $[0, \infty)$ if, for every $T_0 > 0$, $v \in \mathcal{C}([0, T_0]; H)$ and v satisfies the abstract evolution equation (13) in the sense of distributions.

More precisely, for every admissible test function φ , the variational formulation associated with (13) is satisfied on $[0, T_0]$.

With this definition, we state the following well-posedness result:

Theorem 1. Let $v(0) \in H$ and assume that (H1)–(H3) hold. Then problem (13) admits a unique weak solution $v \in \mathcal{C}([0, \infty); H)$. Moreover, if $v(0) \in D(A)$, then

$$v \in \mathcal{C}([0, \infty); D(A)) \cap \mathcal{C}^1([0, \infty); H).$$

Proof. We prove that the operator A is maximal dissipative in H . The Lumer-Phillips theorem then implies that A generates a C_0 -semigroup of contractions on H .

First, we establish the dissipativity of A . Let $v = (y, y_t, z, y(L, t), y_t(L, t))^T \in D(A)$. By the definition of the inner product in H and the operator A , we have

$$\begin{aligned} (Av, v)_H = & \frac{1}{2} \int_0^L y_t \left[(T(r)y_r)_r - (EI(r)y_{rr})_{rr} - \eta_a y_t - \int_{\tau_1}^{\tau_2} \eta_b(s) z(r, 1, t, s) \, ds \right] \, dr \\ & + \frac{1}{2} \int_0^L [EI(r)y_{rrt}y_{rr} + T(r)y_{rt}y_r] \, dr \\ & - \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] \int_0^1 z(r, p, t, s) z_p(r, p, t, s) \, dp \, ds \, dr \\ & + \frac{1}{2} y_t(L, t) \left[(EI(r)q_{rr})_r|_{r=L} - T(L)y_r(L, t) - ky_t(L, t) \right] + \frac{1}{2} yy_t(L, t). \end{aligned}$$

Using integration by parts, the boundary conditions, and the identity (11), we obtain

$$\begin{aligned} (Av, v)_H = & -\frac{1}{2} \left(\eta_a - \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] \, ds \right) \int_0^L y_t^2(r, t) \, dr \\ & - \frac{1}{2} \int_0^L y_t(r, t) \int_{\tau_1}^{\tau_2} \eta_b(s) z(r, 1, t, s) \, ds \, dr \\ & - \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] z^2(r, 1, t, s) \, ds \, dr - \frac{k}{2} y_t^2(L, t) + \frac{1}{2} yy_t(L, t). \end{aligned}$$

Applying result (8), and for the last term using Young's inequality and Poincaré's inequality,

we obtain, for every $\delta > 0$,

$$(Av, v)_H \leq -\frac{1}{2} \left[\eta_a - \int_{\tau_1}^{\tau_2} \left(\eta_b(s) + \frac{\eta_c}{2} \right) ds - \frac{L^2 \delta}{2} \right] \int_0^L y_t^2(r, t) dr \\ - \frac{\eta_c}{2} \int_0^L \int_{\tau_1}^{\tau_2} z^2(r, 1, t, s) ds dr - \frac{k - \delta}{2} y_t^2(L, t).$$

Hence, if $\delta > 0$ is chosen sufficiently small such that $\delta < k$ and $\eta_a > \int_{\tau_1}^{\tau_2} \left(\eta_b(s) + \frac{\eta_c}{2} \right) ds + \frac{L^2 \delta}{2}$, then

$$(Av, v)_H \leq 0, \quad \forall v \in D(A).$$

Thus, A is dissipative.

We next prove that A is maximal ($I - A$ is surjective). Let $G = (g_1, g_2, g_3, g_4, g_5)^T \in H$. We seek $v \in D(A)$ satisfying

$$(I - A)v = G.$$

The corresponding resolvent system is

$$\begin{aligned} y_t &= y - g_1, \\ (\rho A(r) + \eta_a)y_t - (T(r)y_r)_r + (EI(r)y_{rr})_{rr} + \int_{\tau_1}^{\tau_2} \eta_b(s)z(r, 1, t, s) ds &= g_2, \\ z + s^{-1}z_p &= g_3, \\ y_t(L, t) &= y(L, t) - g_4, \\ (M + k)y_t(L, t) - (EI(r)q_{rr})_r|_{r=L} + T(L)y_r(L, t) &= g_5. \end{aligned} \tag{14}$$

The third equation in (14), together with the initial condition $z(r, 0, t, s) = y_t(r, t) = y - g_1$, is a first-order linear ordinary differential equation in p . Its unique solution is

$$\begin{aligned} z(r, p, t, s) &= z(r, 0, t, s)e^{-sp} + se^{-sp} \int_0^p g_3(r, \tau, s)e^{s\tau} d\tau \\ &= \left[(y - g_1) + s \int_0^p g_3(r, \tau, s)e^{s\tau} d\tau \right] e^{-sp}. \end{aligned} \tag{15}$$

In particular, $z(r, 1, t, s) = e^{-s}(y - g_1) + se^{-s} \int_0^1 g_3(r, \tau, s)e^{s\tau} d\tau$.

Substituting the first equation of (14) and the expression above for $z(r, 1, t, s)$ into the second equation, and using the fourth equation in the last equation, we obtain the elliptic problem

$$\begin{aligned} D(r)y - (T(r)y_r)_r + (EI(r)y_{rr})_{rr} &= h_1, \\ (M + k)y(L, t) - (EI(r)q_{rr})_r|_{r=L} + T(L)y_r(L, t) &= h_2, \end{aligned} \tag{16}$$

where $D(r) = \rho A(r) + \eta_a + \int_{\tau_1}^{\tau_2} \eta_b(s)e^{-s} ds$, $h_2 = g_5 + (M + k)g_4$ and $h_1 = \left[\rho A(r) + \eta_a - \int_{\tau_1}^{\tau_2} \eta_b(s)e^{-s} ds \right] g_1 + g_2 - \int_{\tau_1}^{\tau_2} s\eta_b(s)e^{-s} \int_0^1 g_3(r, \tau, s)e^{s\tau} d\tau ds$.

Let $\mathcal{V} = V \times \mathbb{R}$, equipped with the norm

$$\|(y, y(L, t))\|_{\mathcal{V}}^2 = \int_0^L \left[D(r)y^2 + EI(r)y_{rr}^2 + T(r)y_r^2 \right] dr + My^2(L, t).$$

Multiplying the first equation in (16) by a test function $\varphi \in V$, the second by $\varphi(L)$, and integrating by parts, we obtain the variational formulation

$$B((y, y(L, t)), (\varphi, \varphi(L))) = H(\varphi, \varphi(L)), \quad (\varphi, \varphi(L)) \in \mathcal{V},$$

where $H(\varphi, \varphi(L)) = \int_0^L h_1 \varphi dr + h_2 \varphi(L)$, and $B((y, y(L, t)), (\varphi, \varphi(L))) = \int_0^L [D(r)y\varphi + T(r)y_r\varphi_r + EI(r)y_{rr}\varphi_{rr}] dr + (M + k)y(L, t)\varphi(L)$.

Under the assumptions on A , T , η_b , and EI , the bilinear form B is continuous on $\mathcal{V}$, that is

$$B((y, y(L, t)), (y, y(L, t))) = \|(y, y(L, t))\|_{\mathcal{V}}^2 + ky^2(L, t) \leq \left(1 + \frac{k}{M}\right) \|(y, y(L, t))\|_{\mathcal{V}}^2.$$

Moreover,

$$B((y, y(L, t)), (y, y(L, t))) = \|(y, y(L, t))\|_{\mathcal{V}}^2 + ky^2(L, t) \geq \|(y, y(L, t))\|_{\mathcal{V}}^2.$$

Thus, B is coercive on $\mathcal{V}$.

So the required continuity and coercivity estimates hold. The functional H is linear and bounded on $\mathcal{V}$. Therefore, by the Lax-Milgram theorem [11, Corollary 5.8, p. 140], there exists a unique pair $(y, y(L, t)) \in \mathcal{V}$ solving the variational problem. Standard elliptic regularity implies that $v = (y, y_t, z, y(L, t), y_t(L, t))^T \in D(A)$. Hence, $\text{Ran}(I - A) = H$. Therefore, A is maximal dissipative.

Finally, $D(A)$ is dense in H , see [40, Theorem 4.6, p. 16]. Since A is maximal dissipative, the Lumer-Phillips theorem [40, Theorem 4.3, p. 14] implies that A generates a C_0 -semigroup of contractions on H . Consequently, problem (13) is well posed and admits a unique solution. $\square$

4. Stability

Since Lemma 2 and the proposed control (3) do not allow us to determine the sign of the energy derivative (4), we will employ Lyapunov's method to establish energy decay. For this purpose, we introduce the following Lyapunov candidate:

$$\mathcal{L}(t) = E(t) + \beta V(t) + \alpha W(t),$$

where $\beta, \alpha \in \mathbb{R}_+$,

$$V(t) = \rho \int_0^L (A(r)y_t y(r, t)) dr + My_t y(L, t)$$

and

$$W(t) = \int_0^L \int_{\tau_1}^{\tau_2} s[\eta_b(s) + \eta_c] \int_0^1 e^{-2ps} (y_t(r, t - ps))^2 dp ds dr.$$

Proposition 1. There exist positive constants a and b such that

$$aE(t) \leq a(E(t) + \alpha W(t)) \leq \mathcal{L}(t) \leq b(E(t) + \alpha W(t)), \quad \forall t \geq 0. \quad (17)$$

Proof. By Young's inequality, Poincaré's inequality, and assumption (H2), we have

$$\begin{aligned} |V(t)| &\leq \frac{\rho^2}{2} \int_0^L (A(r)y_t(r,t))^2 \, dr + \frac{1}{2} \int_0^L (y(r,t))^2 \, dr + \frac{M^2}{2} (y_t(L,t))^2 + \frac{1}{2} (y(L,t))^2 \\ &\leq \frac{\rho^2 A_{\max}}{2} \int_0^L A(r)y_t^2(r,t) \, dr + \frac{L^2}{T_{\min}} \int_0^L T(r)y_r^2(r,t) \, dr + \frac{M^2}{2} (y_t(L,t))^2 \\ &\leq c_v E(t), \end{aligned}$$

where $c_v = \max\{\rho A_{\max}, 2L^2/T_{\min}, M\}$. Consequently,

$$[1 - \beta c_v] E(t) + \alpha W(t) \leq \mathcal{L}(t) \leq [1 + \beta c_v] E(t) + \alpha W(t).$$

Choose $\beta > 0$ sufficiently small such that $\beta c_v < 1$. Then $a = 1 - \beta c_v > 0$, and $b = 1 + \beta c_v$. Since $a < 1$ and $b > 1$, it follows that

$$aE(t) \leq a(E(t) + \alpha W(t)) \leq \mathcal{L}(t) \leq b(E(t) + \alpha W(t)), \quad t \geq 0.$$

This proves (17). □

Theorem 2. Assume (H1)–(H3) holds. For every initial data $v_0 \in H$, the corresponding energy (5) satisfies

$$E(t) \leq \omega e^{-\lambda t} + \varepsilon, \quad t \geq 0, \quad (18)$$

where ω , λ , and ε are positive constants.

Proof. By differentiating $V(t)$ and using the first equation in system (1), we obtain

$$\begin{aligned} V'(t) &= \rho \int_0^L A(r)(y_t(r,t))^2 \, dr - \int_0^L y(r,t)(EI(r)y_{rr}(r,t))_{rr} \, dr \\ &\quad + \int_0^L y(r,t) \left[(T(r)y_r(r,t))_r - \eta_a y_t(r,t) - \int_{\tau_1}^{\tau_2} \eta_b(s)y_t(r,t-s) \, ds \right] \, dr \\ &\quad + \int_0^L y(r,t)f(r,t) \, dr + M(y_t(L,t))^2 + My(L,t)y_{tt}(L,t). \end{aligned}$$

After integrating by parts and using the boundary conditions in (1), we obtain

$$V'(t) = \rho \int_0^L A(r)(y_t(r,t))^2 \, dr - \int_0^L EI(r)(y_{rr}(r,t))^2 \, dr - \int_0^L T(r)(y_r(r,t))^2 \, dr$$

$$\begin{aligned}
 & - \int_0^L y(r, t) \left[\eta_a y_t(r, t) + \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t - s) \, ds \right] \, dr \\
 & + \int_0^L y(r, t) f(r, t) \, dr + M (y_t(L, t))^2 + y(L, t) (U(t) + g(t)). \quad (19)
 \end{aligned}$$

For the terms with indefinite signs, let $\delta > 0$. Using Young's and Poincaré's inequalities,

$$\begin{aligned}
 & \int_0^L y(r, t) \left[\eta_a y_t(r, t) + \int_{\tau_1}^{\tau_2} \eta_b(s) y_t(r, t - s) \, ds \right] \, dr \leq \frac{L^2 \delta}{2} \left[\eta_a + \left(\int_{\tau_1}^{\tau_2} \eta_b(s) \, ds \right)^2 \right] \int_0^L (y_r(r, t))^2 \, dr \\
 & + \frac{\eta_a}{2\delta} \int_0^L (y_t(r, t))^2 \, dr + \frac{1}{2\delta} \int_0^L \int_{\tau_1}^{\tau_2} \eta_b(s) (y_t(r, t - s))^2 \, ds \, dr. \quad (20)
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 & \int_0^L y(r, t) f(r, t) \, dr \leq \frac{L^2 \delta}{2} \int_0^L (y_r(r, t))^2 \, dr + \frac{1}{2\delta} \int_0^L (f(r, t))^2 \, dr \\
 & \leq \frac{L^2 \delta}{2} \int_0^L (y_r(r, t))^2 \, dr + \frac{\bar{f}}{2\delta}. \quad (21)
 \end{aligned}$$

For the boundary control term, using the proposed control (3), we have

$$y(L, t) U(t) \leq \frac{L^2 \delta}{4} \int_0^L (y_r(r, t))^2 \, dr + \frac{k^2}{L\delta} (y_t(L, t))^2. \quad (22)$$

Similarly,

$$y(L, t) g(t) \leq \frac{L^2 \delta}{4} \int_0^L (y_r(r, t))^2 \, dr + \frac{\bar{g}}{2\delta}. \quad (23)$$

Combining (19)–(23), and using $A(r) \geq A_{\min} > 0$, and $T(r) \geq T_{\min} > 0$, we obtain

$$\begin{aligned}
 V'(t) & \leq \left(\rho + \frac{\eta_a}{2\delta A_{\min}} \right) \int_0^L A(r) (y_t(r, t))^2 \, dr - \int_0^L EI(r) (y_{rr}(r, t))^2 \, dr \\
 & - \left\{ 1 - \frac{L^2 \delta}{2T_{\min}} \left[\eta_a + \left(\int_{\tau_1}^{\tau_2} \eta_b(s) \, ds \right)^2 + 2 \right] \right\} \int_0^L T(r) (y_r(r, t))^2 \, dr \\
 & + \frac{1}{2\delta} \int_0^L \int_{\tau_1}^{\tau_2} \eta_b(s) (y_t(r, t - s))^2 \, ds \, dr + \left(M + \frac{k^2}{L\delta} \right) (y_t(L, t))^2 + \frac{\bar{f} + \bar{g}}{2\delta}. \quad (24)
 \end{aligned}$$

Next, differentiating $W(t)$ and integrating by parts with respect to p , we obtain

$$\begin{aligned}
 W'(t) &= \int_0^L \int_{\tau_1}^{\tau_2} s[\eta_b(s) + \eta_c] \int_0^1 2e^{-2ps} y_t(r, t - ps) y_{tt}(r, t - ps) dp ds dr \\
 &= - \int_0^L \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] \int_0^1 e^{-2ps} \frac{\partial}{\partial p} (y_t(r, t - ps))^2 dp ds dr \\
 &= - \int_0^L \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] e^{-2s} (y_t(r, t - s))^2 ds dr \\
 &\quad + \int_0^L \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] ds (y_t(r, t))^2 dr - 2W(t). \tag{25}
 \end{aligned}$$

Since $s \in [\tau_1, \tau_2]$, we have $e^{-2s} = \frac{e^{-2\tau_2}}{s} s \geq \frac{e^{-2\tau_2}}{\tau_2} s$. Moreover, since $A(r) \geq A_{\min} > 0$, then $(y_t(r, t))^2 \leq \frac{1}{A_{\min}} A(r) (y_t(r, t))^2$. Consequently,

$$\begin{aligned}
 W'(t) &\leq - \frac{e^{-2\tau_2}}{\tau_2} \int_0^L \int_{\tau_1}^{\tau_2} s[\eta_b(s) + \eta_c] (y_t(r, t - s))^2 ds dr \\
 &\quad + \frac{1}{A_{\min}} \left[\int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] ds \right] \int_0^L A(r) (y_t(r, t))^2 dr - 2W(t). \tag{26}
 \end{aligned}$$

Combining the estimates (6), (24), and (26), we obtain

$$\begin{aligned}
 \mathcal{L}'(t) &\leq -\eta_i \int_0^L A(r) (y_t(r, t))^2 dr - \eta_j \int_0^L T(r) (y_r(r, t))^2 dr \\
 &\quad - \beta \int_0^L EI(r) (y_{rr}(r, t))^2 dr - \left(\frac{\eta_c}{2} - \frac{\beta}{2\delta} \right) \int_0^L \int_{\tau_1}^{\tau_2} (y_t(r, t - s))^2 ds dr \\
 &\quad - \frac{e^{-2\tau_2}}{\tau_2} \alpha \int_0^L \int_{\tau_1}^{\tau_2} s[\eta_b(s) + \eta_c] (y_t(r, t - s))^2 ds dr - 2\alpha W(t) - \eta_k (y_t(L, t))^2 + \gamma,
 \end{aligned}$$

where $\eta_i = \frac{\eta_d}{A_{\max}} - \frac{\delta}{2A_{\min}} - \beta \left(\rho + \frac{\eta_a}{2\delta A_{\min}} \right) - \frac{\alpha}{A_{\min}} \int_{\tau_1}^{\tau_2} [\eta_b(s) + \eta_c] ds$, $\eta_k = k - \beta \left(M + \frac{k^2}{2L\delta} \right)$, $\eta_j = \beta \left\{ 1 - \left[\eta_a + \left(\int_{\tau_1}^{\tau_2} \eta_b(s) ds \right)^2 + 2 \right] \frac{L^2 \delta}{2T_{\min}} \right\}$ and $\gamma = \frac{\beta+1}{2\delta} (\bar{f} + \bar{g}) < \infty$.

We have the flexibility to select extremely small values for the parameters β , α , and δ , ensuring that the coefficients in the previous relation become negative. Therefore,

$$\mathcal{L}'(t) \leq -c(E(t) + \alpha W(t)) + \gamma,$$

where $c = 2 \min \left\{ \frac{\eta_i}{\rho}, \eta_j, \beta, \frac{e^{-2\tau_2}}{\tau_2} \alpha, 1, \eta_k \right\} > 0$. Using the equivalence relation (17), we have

$$\mathcal{L}'(t) \leq -\lambda \mathcal{L}(t) + \gamma,$$

where $\lambda = \frac{c}{b} > 0$. Multiplying this inequality by $e^{\lambda t}$ gives

$$\frac{d}{dt}(e^{\lambda t} \mathcal{L}(t)) \leq \gamma e^{\lambda t}.$$

Integrating over $(0, t)$, we obtain

$$\mathcal{L}(t) \leq \left(\mathcal{L}(0) - \frac{\gamma}{\lambda} \right) e^{-\lambda t} + \frac{\gamma}{\lambda} \leq \mathcal{L}(0) e^{-\lambda t} + \frac{\gamma}{\lambda}.$$

Once again, by employing the equivalence proposition (17), we get

$$E(t) \leq \omega e^{-\lambda t} + \varepsilon,$$

where $\omega = \frac{\mathcal{L}(0)}{a}$, and $\varepsilon = \frac{\gamma}{\lambda a}$. □

5. Numerical simulations

To validate the effectiveness of the proposed control law (3), numerical simulations are performed using a finite-difference method on a uniform spatial grid. The spatial derivatives are discretized using centered finite differences, while the time integration employs an explicit second-order central difference scheme. The distributed delay integral is evaluated using the trapezoidal rule with halved endpoint weights. The parameters used in the numerical simulations are listed in Table 1.

A key practical consideration is that high parameter magnitudes demand excessively fine temporal discretization, which significantly raises computational cost. To alleviate this burden in numerical experiments, it is advisable to rescale the model parameters, thereby enhancing simulation efficiency without altering the underlying physical behavior.

The axial tension is considered as $T(r) = 0.1(x + 1)$. The initial data are $y^1(r) = L^2 \left[\cos\left(\frac{r\pi}{2L}\right) - 1 \right]$, $y^2(r) = 0$, and $y^3(r, t) = 0.01 \sin(rt)$.

To investigate the numerical performance of the proposed model, we assume that the distributed load increases linearly along the tower height, from the base $r = 0$ to the top $r = L$.

Table 1. Physical parameters

Parameter	Description	Value
L	tower height	10 m
M	concentrated mass	5 kg
η_a	damping coefficient	0.05 N s m^{-2}
$\rho A(r)$	mass per unit length	$(2 - 0.05r) \text{ kg m}^{-1}$
$EI(r)$	bending stiffness	$(4 - 0.1r) \text{ N m}^2$
$\eta_b(t)$	weight of the delay	$e^{-t/100} \text{ N s m}^{-2}$
τ_1, τ_2	real numbers	0.01, 0.02

Accordingly, the time-dependent distributed excitation is prescribed as

$$f(r, t) = f_0 r \left(\tilde{f} + \sum_{j=1}^4 \sin(\omega_j t) \right),$$

where $\omega_1 = 0.0937$, $\omega_2 = 0.3762$, $\omega_3 = 0.8733$, $\omega_4 = 1.6427$ are the first four natural angular frequencies of the corresponding linearized and undamped tower model [42], obtained numerically from the associated eigenvalue problem. The use of these frequencies in the excitation is intended to examine the response of the tower to harmonic components located at its low-order natural frequencies. Here, $f_0 = \frac{1}{L}$ and $\tilde{f} = 0.1$ denote the amplitude of the oscillating tower of distributed load and the mean level of the distributed excitation.

For the boundary disturbance, we prescribe a time-dependent load containing the same spectral components as the distributed excitation, namely,

$$g(t) = g_0 \left(\tilde{g} + \sum_{j=1}^4 \sin(\omega_j t) \right),$$

where $g_0 = 1$ and $\tilde{g} = 1$ are prescribed amplitude and mean-load parameters, respectively. This choice allows the numerical experiment to account simultaneously for distributed and boundary excitations associated with the dominant low-frequency modes of the tower.

We performed numerical simulations for two cases:

1. *The uncontrolled case:* We first consider the system without boundary control, corresponding to $k = 0$. The evolution of the tower displacement at position r and time t is illustrated in Fig. 1 (left). To provide a more detailed description of the displacement response, we also examine the solution at two observation points, namely, the midpoint ($r = L/2$) and the endpoint ($r = L$). The corresponding time responses are presented in Figs. 2 and 3, respectively.
2. *The controlled case:* We then activate the boundary control at the endpoint ($r = L$), with the control gain set to $k = 3$. The resulting controlled displacement evolution is shown in

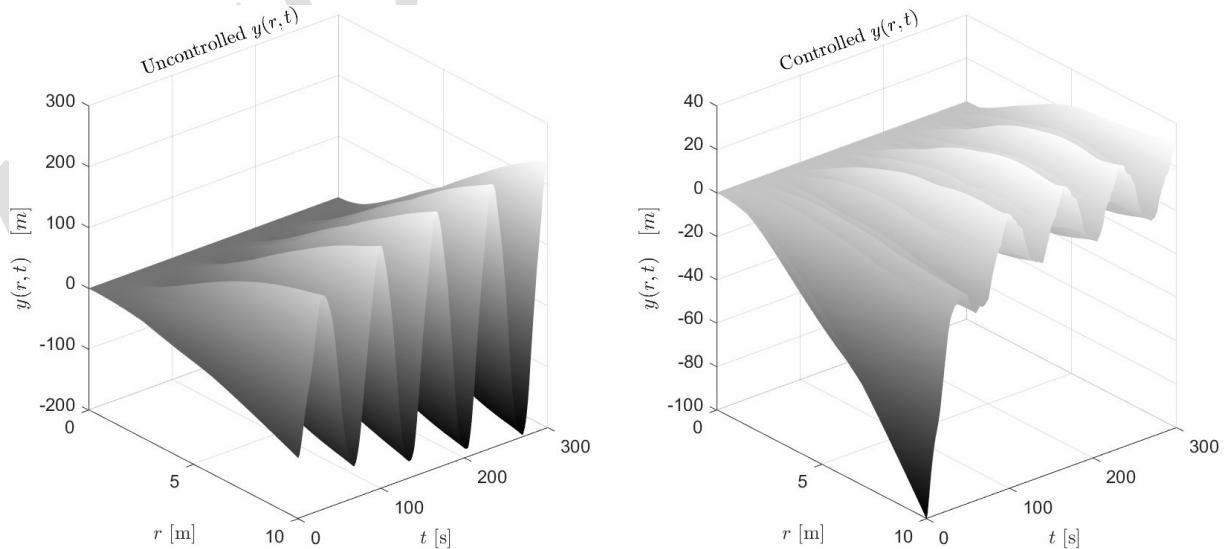


Fig. 1. Uncontrolled (left) and controlled (right) tower vibration $y(r, t)$

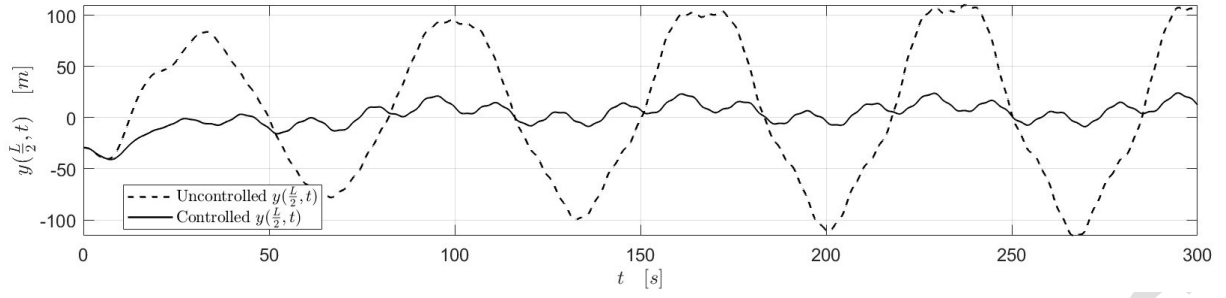


Fig. 2. Uncontrolled/controlled tower vibration at $r = L/2$

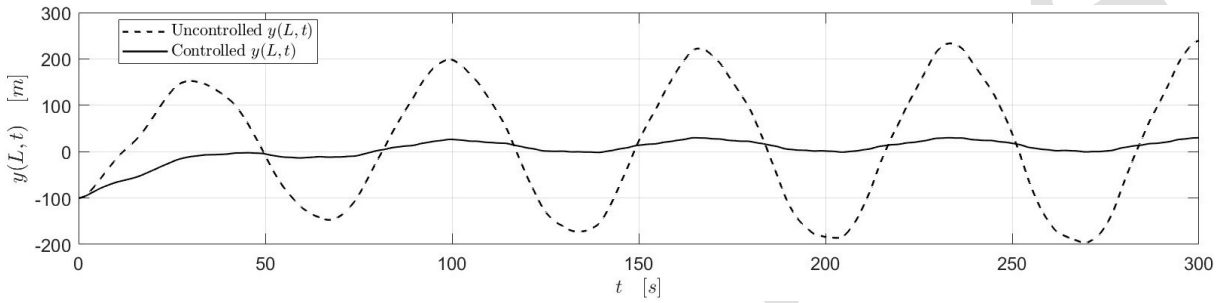


Fig. 3. Uncontrolled/controlled tower vibration at $r = L$

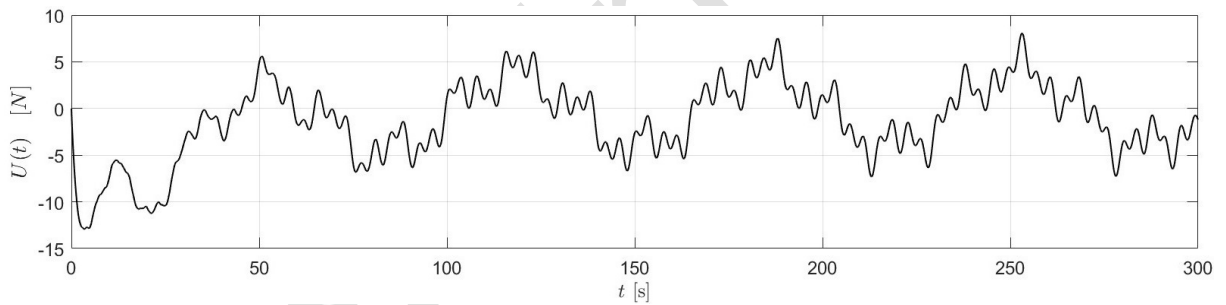


Fig. 4. Boundary control $U(t)$

Fig. 1 (right). For comparison with the uncontrolled case, the displacement responses at the two observation points $r = L/2$ and $r = L$ are displayed in Figs. 2 and 3, respectively. Fig. 4 presents the corresponding boundary control applied at $r = L$.

To further assess the effectiveness of the proposed control strategy, Fig. 5 compares the evolution of the system energy in the controlled and uncontrolled cases. In the same figure, we also plot the reference function $\omega e^{-\lambda t} + \varepsilon$, which is used to assess the theoretical decay estimate established in (18). This comparison provides a quantitative evaluation of the damping performance of the boundary control and illustrates the agreement between the numerical results and the theoretical energy decay behavior.

These figures clearly depict significant vibrations of the tower before the control is activated, which gradually diminish after the control is applied. This illustrates the effectiveness of the control (3) in rapidly reducing tower vibrations and stabilizing it near the zero position. These observations align with the results of Theorem 2.

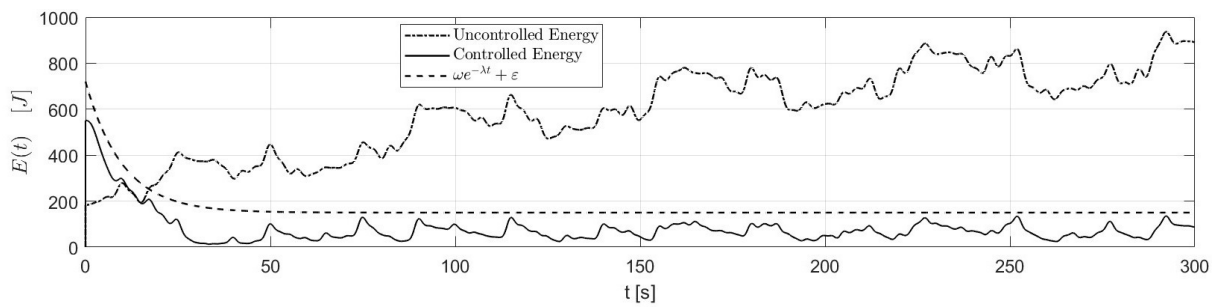


Fig. 5. Uncontrolled/controlled energy tower vibration

6. Conclusions

This study successfully designs a boundary control strategy for a wind turbine tower modeled as a non-uniform Euler-Bernoulli beam with distributed delay. Using semi-group theory for well-posedness and Lyapunov analysis for stability, the controller ensures the tower's vibrations converge to a small neighborhood of zero. Numerical simulations confirm the effectiveness and robustness of the proposed approach.

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