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Stress analysis of finite length cylinders of layered media

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Abstract

In this paper, we analyze an orthotropic, layered $(0°/90°)$ and (0°/core/0°) sandwich cylinders under pressurized load with a diaphragm supported boundary conditions which is considered as a two dimensional (2D) plane strain boundary value problem of elasticity in (r, z) direction. A simplified numerical cum analytical approach is used for the analysis. Boundary conditions are satisfied exactly by using an analytical expression in longitudinal (z) direction in terms of Fourier series expansion. Resulting first order simultaneous ordinary differential equations (ODEs) with boundary conditions prescribed at $r = r_i$, r_o defines a two point boundary value problem (BVP), whose equations are integrated in radial direction through an effective numerical integration technique by first transforming the BVP into a set of initial value problems (IVPs). Numerical solutions are first validated for their accuracy with 1D solution of an infinitely long cylinder. Stresses and displacements in cylinders of finite lengths having various l/R and h/R ratios are presented for future reference.

Keywords: elasticity theory, circular cylinder, numerical integration, boundary value problems, laminated composites

1. Introduction

Composites have seen an ever increasing use in the process industry during the last twenty five years. Their use as a material of choice for pressure vessels and components is due to the fact that they possess longer life in a corrosive environment, low weight but high strength and stiffness, and the capability to tailor directional strength properties to design needs. Composite cylinders are widely used in various engineering applications such as aerospace vehicles, nuclear pressure vessels, piping and many other engineering structures and need accurate analysis of deformations and stresses induced by applied pressure loading. The classic problem of an infinitely long elastic cylinder of an isotropic material under internal and external pressure was analyzed first by Lame in 1847 (given in [16]) for isotropic and by [12] for anisotropic and layered materials. This particular problem has been studied by many during later years. In paper [9], authors obtained stresses and displacements by the use of three dimensional (3D) elasticity theory and several shell theories in a long isotropic circular cylinder subjected to an axisymmetric radial line load and compared results with the shell theories of Love and Flugge. An elasticity solution by using a Love function approach for semi-infinite circular cylindrical shell subjected to a concentrated axisymmetric radial line load at the free end was presented in [3]. The problem of an infinite circular cylindrical shell subjected to periodically spaced band loads using 3D elasticity theory and the shell theories of Love (and Donnell), Flugge, and a theory developed by Reissner and Nagdhi was solved in [10]. An approximate solution to the Navier equations of the 3D elasticity for an axisymmetric orthotropic infinitely long circular

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cylinder subjected to internal and external pressure, axial loads, and closely spaced periodic radial loads was obtained in [13]. An exact solution for a thick, transversely isotropic, simply supported finite length circular cylindrical shell subjected to axisymmetric load using a transfer matrix approach are obtained in [1]. Clamped-clamped and clamped-simply supported cylindrical shells by a so-called segmentation numerical integration technique was analyzed by [8]. The same technique for elastic analysis of cylindrical pressure vessels with various end closures using Love's classical shell theory used in [15].

In this paper, governing differential equations from theory of 3D anisotropic elasticity, which govern the behaviors of a finite length circular orthotropic cylinder in a state of symmetric plane strain in (r, z) under sinusoidal pressurized loading which is a function of both radial and axial coordinates, are taken. By assuming a global analytical solution in the longitudinal direction (z) which satisfies the two end boundary conditions exactly, dimensional reduction is done with this process, the 2D generalized plane strain problem is reduced to a 1D problem in the radial coordinate. The equations are reformulated to enable application of an efficient and accurate numerical integration technique developed and proposed for the solution of BVP [7].

In addition, one dimensional elasticity equations of an infinitely long symmetric cylinder are utilized to formulate the mathematical model suitable for numerical integration. These equations are summarized in the Appendix. This has been done with a view to check and compares the results of the present formulation of finite length cylinder under uniform internal/external pressure load, when the length of the cylinder tends to infinity.

The basic governing equations

Basic governing equations of a symmetric cylinder which is considered plane strain in (r, z) direction [12] in cylindrical coordinates (Fig. 1) is written as:

Equilibrium equations

$$\begin{aligned}\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}&=0,\\ \frac{\partial\tau_{zr}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{zr}}{r}&=0.\end{aligned}\tag{1a}$$

Strain displacement relations

$$\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{u}{r},\quad \varepsilon_z=\frac{\partial w}{\partial z},\quad \gamma_{zr}=\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}.\tag{1b}$$

Stress-strains relations for cylindrically orthotropic material

$$\begin{aligned}\varepsilon_r&=\frac{\sigma_r}{E_r}-\nu_{\theta r}\frac{\sigma_\theta}{E_\theta}-\nu_{zr}\frac{\sigma_z}{E_z},\\ \varepsilon_\theta&=-\nu_{r\theta}\frac{\sigma_r}{E_r}+\frac{\sigma_\theta}{E_\theta}-\nu_{z\theta}\frac{\sigma_z}{E_z},\\ \varepsilon_z&=-\nu_{rz}\frac{\sigma_r}{E_r}-\nu_{\theta z}\frac{\sigma_\theta}{E_\theta}+\frac{\sigma_z}{E_z},\quad \gamma_{rz}=\frac{\tau_{rz}}{G_{rz}}.\end{aligned}\tag{1c}$$

Stresses in terms of strains can be written as follows:

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\ C_{21}&C_{22}&C_{23}\\ C_{31}&C_{32}&C_{33}\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \varepsilon_z\end{Bmatrix},\quad \tau_{rz}=G\gamma_{rz}\tag{1d}$$

in which

$$\nu_{r\theta} = \frac{\nu_{\theta r}}{E_\theta} E_r, \quad \nu_{rz} = \frac{\nu_{zr}}{E_z} E_r, \quad \nu_{z\theta} = \frac{\nu_{\theta z}}{E_\theta} E_z,$$
$$C_{11} = \frac{E_r(1 - \upsilon_{\theta z}\upsilon_{z\theta})}{\Delta}, \quad C_{12} = \frac{E_r(\upsilon_{\theta r} + \upsilon_{zr}\upsilon_{\theta z})}{\Delta}, \quad C_{13} = \frac{E_r(\upsilon_{zr} + \upsilon_{\theta r}\upsilon_{z\theta})}{\Delta},$$
$$C_{22} = \frac{E_\theta(1 - \upsilon_{rz}\upsilon_{zr})}{\Delta}, \quad C_{23} = \frac{E_\theta(\upsilon_{z\theta} + \upsilon_{r\theta}\upsilon_{zr})}{\Delta}, \quad C_{33} = \frac{E_z(1 - \upsilon_{r\theta}\upsilon_{\theta r})}{\Delta}, \tag{1e}$$

where $\Delta = (1 - \nu_{r\theta}\nu_{\theta r} - \nu_{\theta z}\nu_{z\theta} - \nu_{zr}\nu_{rz} - 2\nu_{\theta r}\nu_{z\theta}\nu_{rz})$, $C_{21} = C_{12}$, $C_{32} = C_{23}$, $C_{31} = C_{13}$. Stresses in terms of displacement components can be cast as follows:

$$\sigma_r = C_{11}\left(\frac{\partial u}{\partial r}\right) + C_{12}\left(\frac{u}{r}\right) + C_{13}\left(\frac{\partial w}{\partial z}\right), \ \sigma_\theta = C_{21}\left(\frac{\partial u}{\partial r}\right) + C_{22}\left(\frac{u}{r}\right) + C_{23}\left(\frac{\partial w}{\partial z}\right), \tag{1f}$$
$$\sigma_z = C_{31}\left(\frac{\partial u}{\partial r}\right) + C_{32}\left(\frac{u}{r}\right) + C_{33}\left(\frac{\partial w}{\partial z}\right), \ \tau_{rz} = G\gamma_{rz} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)$$

and boundary conditions in the longitudinal and radial directions are

$$u = \sigma_z = 0 \text{ for } z = 0, l\,; \quad \sigma_r = \tau_{rz} = 0 \text{ for } r = r_i\,; \quad \sigma_r = -p(z)\,, \quad \tau_{rz} = 0 \text{ for } r = r_0 \tag{2}$$

in which l is the length, r_i is the inner radius and r_o is the outer radius of a hollow cylinder.

Load $p(z)$ can be represented in terms of Fourier series in general form as follows:

$$p(z) = \sum_{i=1,3,5,\dots}^{N} p_i \sin\frac{i\pi z}{l} \tag{3a}$$

in which p_i is the Fourier load coefficient which can be determined by using the orthogonality conditions and for sinusoidal loading

$$p(z) = p_0 \sin\frac{\pi z}{l}, \tag{3b}$$

p_0 is the maximum intensity of distributed pressure. The positive coordinates and loadings on a cylinder are shown in Fig. 1a, b.

2. Mathematical formulation

Radial direction r is chosen to be a preferred independent coordinate. Four fundamental dependent variables, displacements u and w and corresponding stresses σ_r and τ_{rz} that occur naturally on a tangent plane $r = \text{constant}$, are chosen in the radial direction. Circumferential stress σ_θ and axial stress σ_z are treated here as auxiliary variables since these are found to be dependent on the chosen fundamental variables [16]. A set of four first order partial differential equations in independent coordinate r which involve only fundamental variables is obtained through algebraic manipulation of Eqs. (1a)–(1f). These are

$$\frac{\partial u}{\partial r} = \frac{\sigma_r}{C_{11}} - \frac{C_{12}}{C_{11}}\left(\frac{u}{r}\right) - \frac{C_{13}}{C_{11}}\left(\frac{\partial w}{\partial z}\right),$$
$$\frac{\partial w}{\partial r} = \frac{1}{G}\tau_{rz} - \frac{\partial u}{\partial z}, \tag{4a}$$
$$\frac{\partial \sigma_r}{\partial r} = -\frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r}{r}\left(\frac{C_{21}}{C_{11}} - 1\right) - \left(\frac{C_{21}C_{12}}{C_{11}} - C_{22}\right)\left(\frac{u}{r^2}\right) - \left(\frac{C_{21}C_{13}}{C_{11}} - C_{23}\right)\left(\frac{1}{r}\frac{\partial w}{\partial z}\right),$$
$$\frac{\partial \tau_{rz}}{\partial r} = -\frac{\tau_{rz}}{r} - \frac{C_{31}}{C_{11}}\frac{\partial \sigma_r}{\partial z} - \left(C_{32} - \frac{C_{12}C_{31}}{C_{11}}\right)\frac{\partial}{\partial z}\left(\frac{u}{r}\right) - \left(C_{33} - \frac{C_{13}C_{31}}{C_{11}}\right)\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right)$$

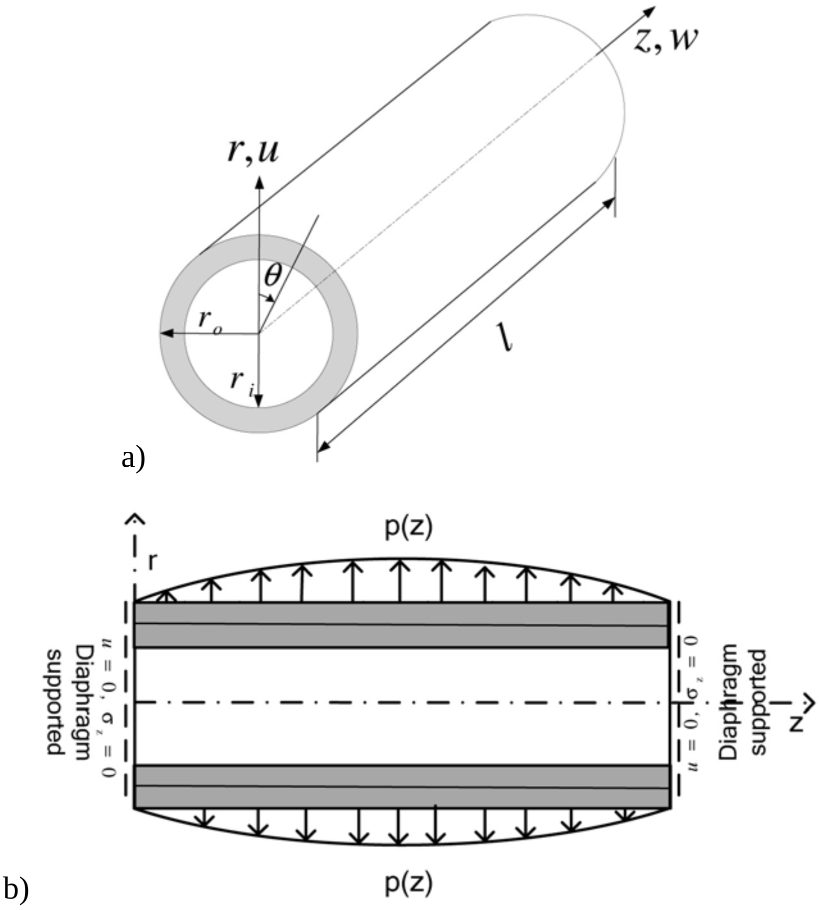


Fig. 1. a) Coordinate system and geometry of cylinder, b) finite cylinder under sinusoidal external pressure loading

and the auxiliary variables

$$\sigma_\theta = C_{21}\left(\frac{\partial u}{\partial r}\right) + C_{22}\left(\frac{u}{r}\right) + C_{23}\left(\frac{\partial w}{\partial z}\right), \quad \sigma_z = C_{31}\left(\frac{\partial u}{\partial r}\right) + C_{32}\left(\frac{u}{r}\right) + C_{33}\left(\frac{\partial w}{\partial z}\right). \tag{4b}$$

Variations of the four fundamental dependent variables which completely satisfy the boundary conditions of simple (diaphragm) supports at $z = 0, l$ can then be assumed as

$$\begin{aligned} u(r,z) &= U(r)\sin\frac{\pi z}{l}, \quad w(r,z) = W(r)\cos\frac{\pi z}{l}, \\ \sigma_r(r,z) &= \sigma(r)\sin\frac{\pi z}{l}, \quad \tau_{rz}(r,z) = \tau(r)\cos\frac{\pi z}{l}. \end{aligned} \tag{5}$$

Substitution of Eq. (5) in Eq. (4a) and simplification, resulting from orthogonality conditions of trigonometric functions, leads to the following four simultaneous ordinary differential equations involving only fundamental variables. These are

$$\begin{aligned} U'(r) &= \frac{\sigma(r)}{C_{11}} - \frac{C_{12}}{C_{11}}\left(\frac{U(r)}{r}\right) + \frac{C_{13}}{C_{11}}\left(\frac{\pi}{l}W(r)\right), \\ W'(r) &= \frac{1}{G}\tau(r) - U(r)\frac{\pi}{l}, \end{aligned}$$

$$\sigma'(r) = \frac{\pi}{l}\tau(r) + \left(\frac{C_{21}}{C_{11}} - 1\right)\frac{\sigma(r)}{r} - \left(\frac{C_{21}\cdot C_{12}}{C_{11}} - C_{22}\right)\left(\frac{U(r)}{r^2}\right) + \left(\frac{C_{21}C_{13}}{C_{11}} - C_{23}\right)\left(\frac{\pi}{l}\frac{W(r)}{r}\right),$$

$$\tau'(r) = -\frac{\tau(r)}{r} - \frac{\pi}{l}\frac{C_{31}}{C_{11}}\sigma(r) - \left(C_{32} - \frac{C_{12}C_{31}}{C_{11}}\right)\left(\frac{\pi}{l}\frac{U(r)}{r}\right) + \left(C_{33} - \frac{C_{13}C_{31}}{C_{11}}\right)\left(\left(\frac{\pi}{l}\right)^2 W(r)\right) \tag{6a}$$

and the auxiliary variables

$$\sigma_\theta = \left[\frac{C_{21}}{C_{11}}\sigma(r) - \left(\frac{C_{21}C_{12}}{C_{11}} - C_{22}\right)\left(\frac{U(r)}{r}\right) + \left(\frac{C_{13}C_{21}}{C_{11}} - C_{23}\right)\left(\frac{\pi}{l}W(r)\right)\right]\sin\frac{\pi z}{l},$$

$$\sigma_z = \left[\frac{C_{31}}{C_{11}}\sigma(r) - \left(\frac{C_{31}C_{12}}{C_{11}} - C_{32}\right)\left(\frac{U(r)}{r}\right) + \left(\frac{C_{13}C_{31}}{C_{11}} - C_{33}\right)\left(\frac{\pi}{l}W(r)\right)\right]\sin\frac{\pi z}{l}. \tag{6b}$$

3. Solution

The above system of first order simultaneous ordinary differential equations (Eq. (6a)) together with the appropriate boundary conditions at the inner and outer edges of the cylinder (Eq. (2)) forms a two-point BVP. However, a BVP in ODEs cannot be numerically integrated as only a half of the dependent variables (two) are known at the initial edge and numerical integration of an ODE is intrinsically an IVP. It becomes necessary to transform the problem into a set of IVPs. The initial values of the remaining two fundamental variables must be selected so that the complete solution satisfies the two specified conditions at the terminal boundary [7]. This technique has been successfully applied to the solutions of plate's problems [4–6, 8, 15]. However, this approach was not used for cylindrical problems in that literature. Runge-Kutta fourth order algorithm with modifications suggested by Gill [2] is used for the numerical integration of the IVPs. A computer code in FORTRAN 77 was written to perform the numerical integration.

4. Numerical Results

Nondimensionalized parameters are defined for pressure loading as follows:

$$\overline{r} = \frac{r}{R}, \quad (\overline{u}, \overline{w}) = \frac{E_r}{pR}(u, w), \quad (\overline{\sigma_r}, \overline{\sigma_\theta}, \overline{\sigma_z}, \overline{\tau_{rz}}) = \frac{1}{p}(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}).$$

Following material properties [11] are taken for orthotropic ($0°$) for Graphite-epoxy material-fibers are oriented in circumferential direction and layered ($0°/90°$) cylinders.

Layer-1 (fibers are oriented in circumferential direction 0-degree)

$$E_r = 9.65\times 10^6, \quad E_\theta = 148\times 10^6, \quad E_z = 9.65\times 10^6, \quad G_{zr} = 3.015\times 10^6,$$
$$\nu_{\theta r} = 0.3, \quad \nu_{zr} = 0.6, \quad \nu_{\theta z} = 0.3.$$

Layer-2 (fibers are oriented in axial direction 90-degree)

$$E_r = 9.65\times 10^6, \quad E_\theta = 9.65\times 10^6, \quad E_z = 148\times 10^6, \quad G_{zr} = 4.55\times 10^6,$$
$$\nu_{\theta r} = 0.6, \quad \nu_{zr} = 0.3, \quad \nu_{\theta z} = 0.0195.$$

Following material properties [14] are taken for the (0°/core/0°) sandwich cylinder:

Face Material properties are

$$E_r = 6.894 \times 10^6, \quad E_\theta = 172.36 \times 10^6, \quad E_z = 6.894 \times 10^6, \quad G_{zr} = 1.378 \times 10^6,$$
$$\nu_{\theta r} = 0.25, \quad \nu_{zr} = 0.25, \quad \nu_{\theta z} = 0.25.$$

Core Material properties are

$$E_r = 3.44 \times 10^6, \quad E_\theta = 0.275 \times 10^6, \quad E_z = 0.275 \times 10^6, \quad G_{zr} = 0.413 \times 10^6,$$
$$\nu_{\theta r} = 0.019\,9, \quad \nu_{zr} = 0.019\,9, \quad \nu_{\theta z} = 0.25.$$

Table 1. Non-dimensional radial stress, radial displacement and hoop stress for simple diaphragm supported orthotropic composite cylinder for $h/R = 1/5$, $1/20$, $1/50$

Quantity	h/R	$\overline{r}$	Present – Finite Length Cylinder			Analytical elasticity solution Lekhnitskii [12]	Numerical solution for infinitely long cylinder
			l/R				
			1	4	100–200		
$\overline{\sigma_r}$ $(z = l/2)$	1/5	0.9	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.526 3	0.532 0	0.532 4	0.537 1	0.537 1
		1.1	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/20	0.975	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.514 1	0.515 3	0.515 3	0.516 4	0.515 3
		1.02 5	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/50	0.99	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.506 2	0.506 7	0.506 7	0.507 1	0.507 1
		1.01	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
$\overline{u}$ $(z = l/2)$	1/5	0.9	0.299 5	0.316 2	0.317 0	0.323 1	0.323 1
		1	0.334 4	0.343 5	0.343 8	0.340 5	0.340 5
		1.1	0.384 5	0.400 3	0.401 1	0.406 6	0.406 6
	1/20	0.975	1.321 5	1.326 2	1.326 4	1.327 9	1.326 4
		1	1.322 7	1.325 1	1.325 1	1.324 3	1.325 1
		1.025	1.327 1	1.331 8	1.332 0	1.333 4	1.332 0
	1/50	0.99	3.286 7	3.288 6	3.288 7	3.289 3	3.289 3
		1	3.281 2	3.282 2	3.282 2	3.281 9	3.281 9
		1.01	3.277 1	3.279 0	3.279 0	3.279 6	3.279 6
$\overline{\sigma_\theta}$ $(z = l/2)$	1/5	0.9	4.925 3	5.284 6	5.304 3	5.506 2	5.304 3
		1	5.279 7	5.426 6	5.430 6	5.383 4	5.430 6
		1.1	5.831 5	5.973 3	5.978 0	5.969 3	5.978 0
	1/20	0.975	20.598 4	20.764 3	20.773 7	20.887 5	20.773 7
		1	20.439 0	20.476 7	20.477 6	20.465 5	20.477 6
		1.025	20.340 3	20.319 2	20.316 1	20.250 9	20.316 1
	1/50	0.99	50.730 3	50.851 0	50.858 4	50.956 3	50.858 4
		1	50.475 3	50.490 3	50.490 6	50.485 8	50.490 6
		1.01	50.246 5	50.183 8	50.179 0	50.100 5	50.179 0

Table 2. Non-dimensional radial stress, radial displacement and hoop stress for simple diaphragm supported orthotropic laminated (0°/90°) composite cylinder for $h/R = 1/5$, $1/20$, $1/50$

Quantity	h/R	$\overline{r}$	Present – Finite Length Cylinder, l/R = 1	4	100–200	Analytical elasticity solution Lekhnitskii [12]	Numerical solution for infinitely long cylinder
$\overline{\sigma_r}$ $(z = l/2)$	1/5	0.9	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.924 8	0.983 3	0.982 8	0.982 8	0.980 1
		1.1	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/20	0.975	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.947 4	0.951 5	0.951 4	0.951 4	0.951 4
		1.025	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/50	0.99	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.943 9	0.944 3	0.944 3	0.944 3	0.944 3
		1.01	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
$\overline{u}$ $(z = l/2)$	1/5	0.9	0.543 8	0.593 3	0.593 0	0.591 3	0.593 0
		1	0.577 3	0.601 7	0.599 1	0.591 4	0.599 1
		1.1	0.605 1	0.631 0	0.628 8	0.620 3	0.628 8
	1/20	0.975	2.431 6	2.435 8	2.434 9	2.446 2	2.434 9
		1	2.420 0	2.416 3	2.414 8	2.397 8	2.414 8
		1.025	2.400 1	2.396 9	2.395 5	2.378 0	2.395 5
	1/50	0.99	6.103 8	6.094 7	6.093 6	6.124 5	6.119 9
		1	6.082 3	6.070 0	6.068 7	6.033 4	6.106 2
		1.01	6.052 7	6.040 7	6.039 4	6.003 7	6.076 0
$\overline{\sigma_\theta}$ $(z = l/2)$	1/5	0.9	9.085 5	10.131 4	10.144 4	10.076 0	10.144 4
		1	1.063 5	1.178 5	1.181 1	1.181 1	1.181 1
		1.1	1.220 5	1.172 4	1.163 9	1.163 9	1.163 9
	1/20	0.975	38.175 3	38.485 2	38.486 0	38.478 8	38.486 0
		1	2.893 2	2.965 1	2.968 5	2.968 6	2.968 5
		1.025	3.022 8	2.927 6	2.919 9	2.920 0	2.919 9
	1/50	0.99	94.773 9	94.878 1	94.877 0	94.879 6	94.877 0
		1	6.533 3	6.595 7	6.599 3	6.600 0	6.599 3
		1.01	6.657 3	6.551 2	6.543 6	6.544 3	6.543 6

Radial and hoop quantities are maximum at $z = l/2$ whereas axial quantities are maximum at $z = 0$, l. Analytical solution for radial stress, hoop stress and radial displacement from exact theory of anisotropic elasticity for infinitely long plane strain cylinder is given in Lekhnitskii [12]. These are used to validate and check the present results throughout wherever applicable. Comparisons of the results are given in Tables 1, 2 and 3.

Three sets of numerical results are presented in the above tables, i.e., results from the present finite length cylinder formulation, computations on the analytical formulae available for infinitely long cylinder [12] and numerically integrated values of the BVP of the infinitely long cylinder (see Appendix).

Here, first a long cylinder is subjected to a sinusoidal pressure load; the results within the limited central length zone only are compared with the plane strain one dimensional solutions.

Table 3. Non-dimensional radial stress, radial displacement and hoop stress for simple diaphragm supported sandwich composite cylinder for $h/R = 1/5$, $1/20$, $1/50$

Quantity	h/R	$\bar{r}$	Present – Finite Length Cylinder l/R = 1	4	100–200	Analytical elasticity solution Lekhnitskii [12]	Numerical solution for infinitely long cylinder
$\overline{\sigma_r}$ $(z = l/2)$	1/5	0.9	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.548 2	0.552 8	0.553 1	0.553 1	0.553 1
		1.1	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/20	0.975	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.520 2	0.521 3	0.521 4	0.521 4	0.521 4
		1.025	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
	1/50	0.99	0.000 0	0.000 0	0.000 0	0.000 0	0.000 0
		1	0.508 7	0.509 1	0.509 2	0.509 2	0.509 4
		1.01	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
$\overline{u}$ $(z = l/2)$	1/5	0.9	0.967 4	0.996 4	0.996 7	0.996 2	0.996 5
		1	1.060 6	1.087 0	1.087 0	1.134 0	1.086 9
		1.1	1.149 8	1.179 3	1.179 6	1.179 9	1.180 2
	1/20	0.975	4.040 7	4.049 6	4.049 7	4.049 0	4.049 7
		1	4.057 4	4.065 1	4.065 1	4.109 5	4.065 1
		1.025	4.073 8	4.082 7	4.082 8	4.082 9	4.082 8
	1/50	0.99	10.022 4	10.025 8	10.025 8	10.025 2	10.025 8
		1	10.024 9	10.027 8	10.027 8	10.071 0	10.027 8
		1.01	10.027 4	10.030 8	10.030 8	10.030 9	10.030 8
$\overline{\sigma_\theta}$ $(z = l/2)$	1/5	0.9	26.616 4	27.621 1	27.647 8	27.673 7	27.647 8
		1	0.053 9	0.056 2	0.056 2	0.056 2	0.056 2
		1.1	26.599 8	27.088 3	27.081 2	27.068 6	27.081 2
	1/20	0.975	103.329 3	103.791 4	103.809 4	103.826 8	103.809 4
		1	0.173 8	0.174 3	0.174 3	0.174 3	0.174 3
		1.025	99.877 7	99.867 2	99.853 8	99.839 7	99.853 8
	1/50	0.99	252.821 8	253.143 0	253.159 4	253.175 4	253.159 4
		1	0.411 6	0.411 9	0.411 9	0.411 9	0.411 9
		1.01	248.728 1	248.584 2	248.569 6	248.555 1	248.569 6

A good agreement is obtained. It is clearly seen that for long cylinders with higher l/R ratios, the results are close to the elasticity solution given by Lekhnitskii [12], for thick, moderately thick and thin cases.

Figs. 2–4 show the through thickness variation of basic as well as auxiliary hoop quantities for orthotropic cylinder for various l/R and h/R ratios. It is seen from Fig. 2 that radial stress varies linearly through thickness; radial displacement is linear for thick orthotropic cylinder whereas reverse trend is seen in thin cylinder. From Fig. 3 it can be seen that hoop stress varies parabolically in case of thick cylinder whereas it varies linearly in case of thin orthotropic cylinder. Parabolic variation of shear stress is seen in Fig. 4, axial displacement is constant through thickness as seen in Fig. 4. Numerical results for laminated ($0°/90°$) and sandwich ($0°$/core/$0°$) cylinder are presented in Figs. 5–7 and Figs. 8–9.

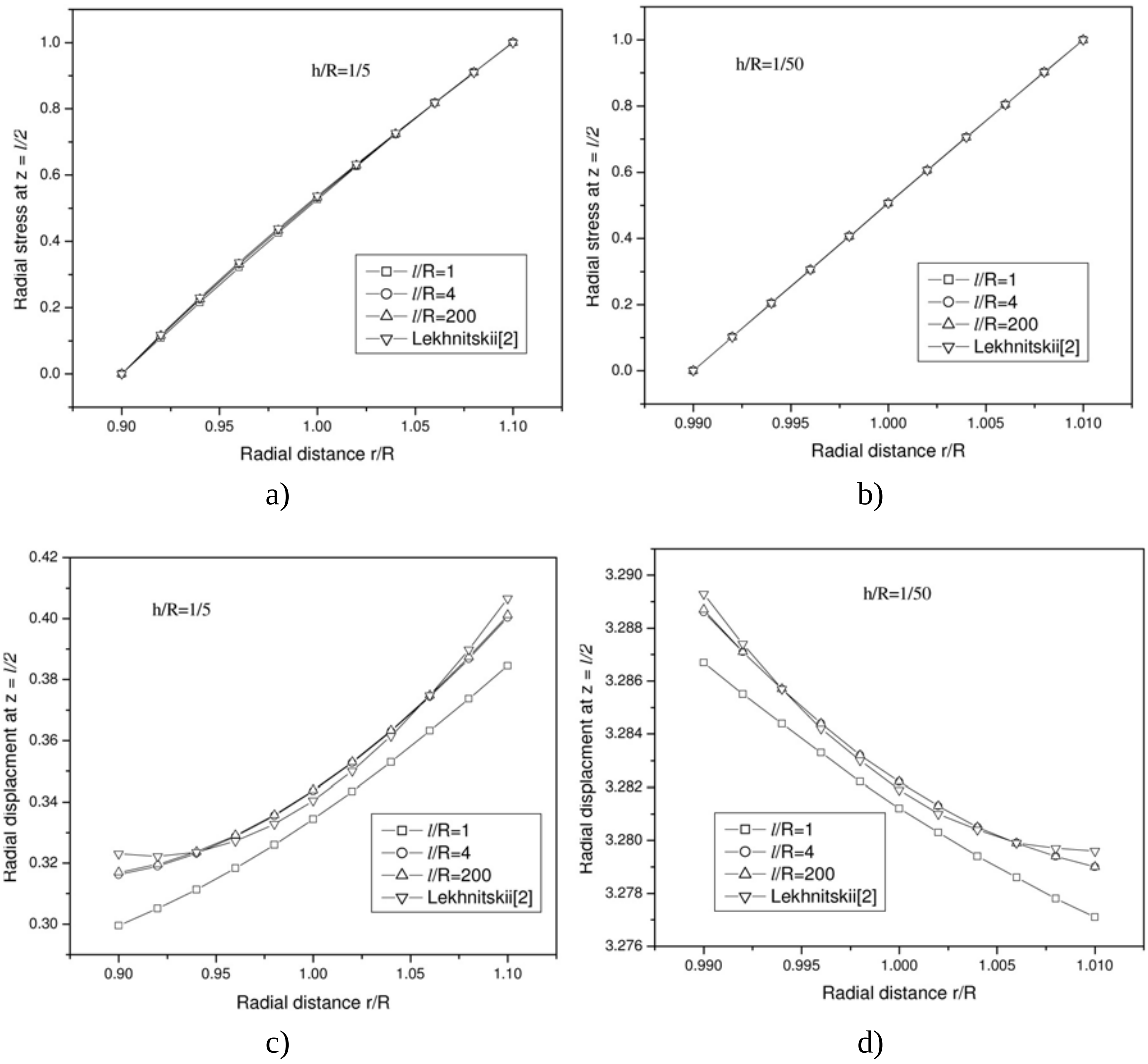


Fig. 2. Distribution of radial stress $\overline{\sigma_r}$ and radial displacement $\overline{u}$ through thickness subjected to sinusoidal loading for orthotropic cylinder

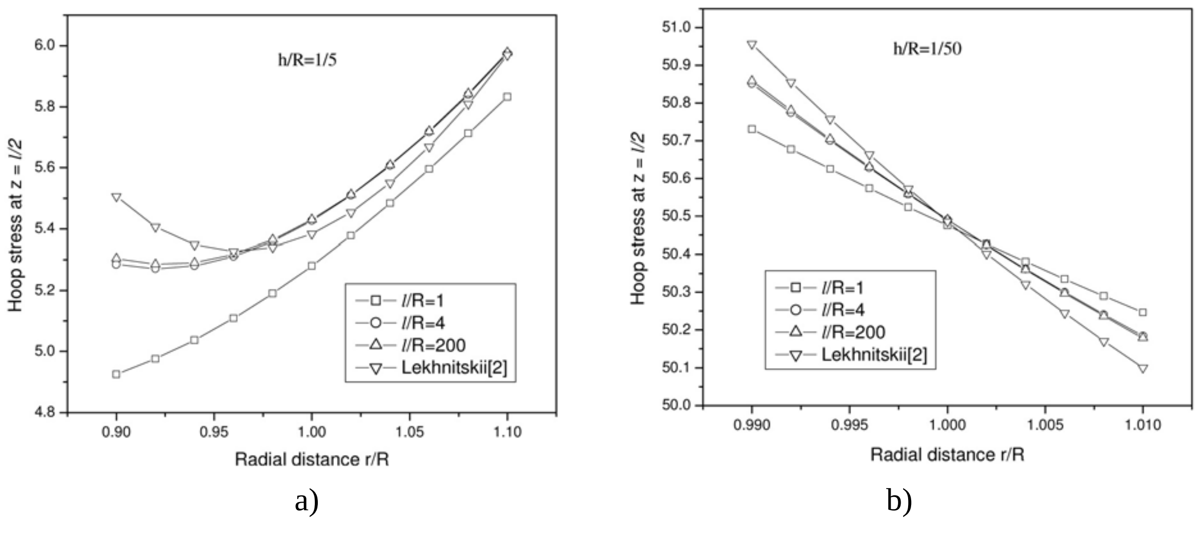
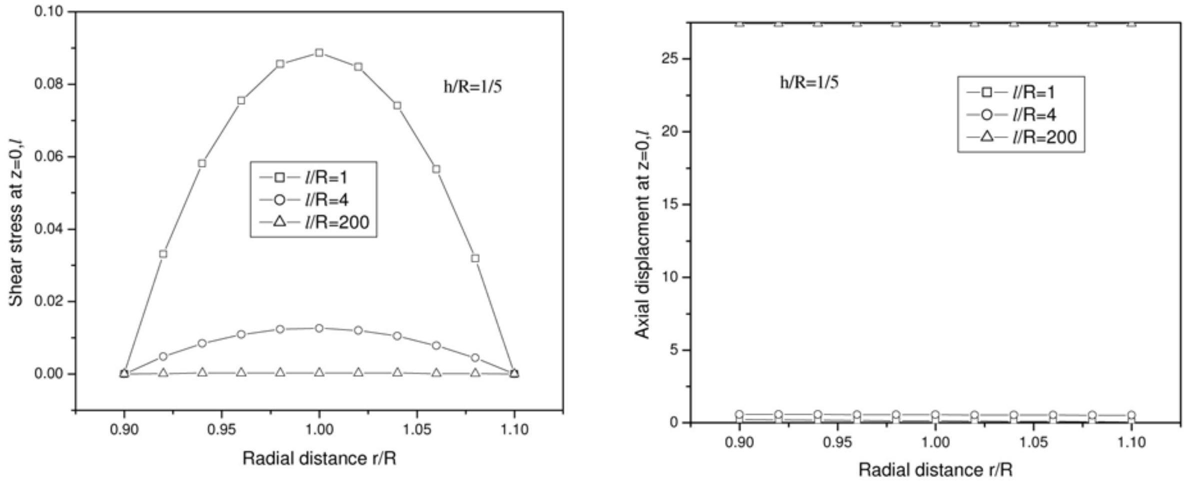
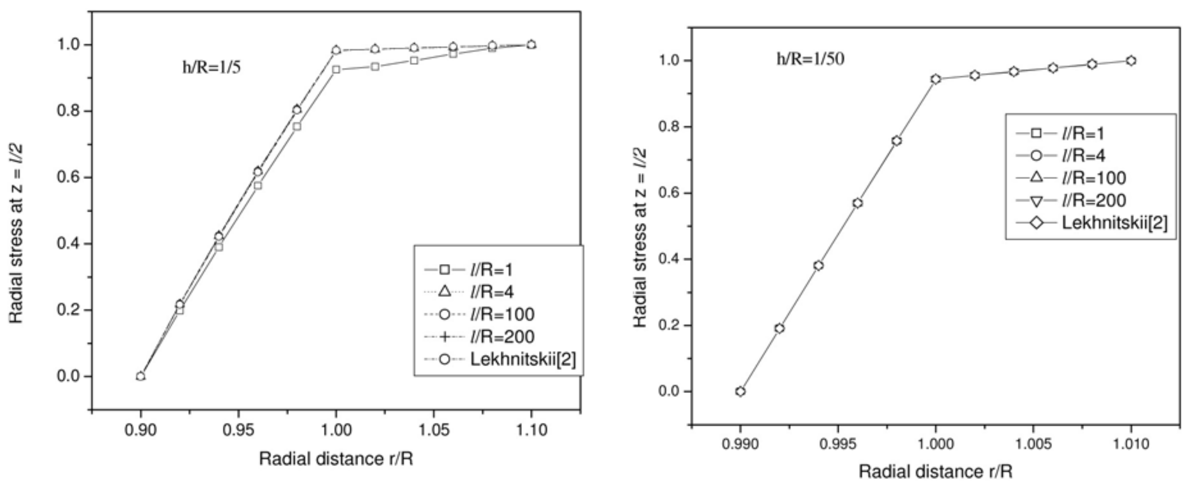


Fig. 3. Distribution of hoop stress $\overline{\sigma_\theta}$ through thickness subjected to sinusoidal loading for orthotropic cylinder

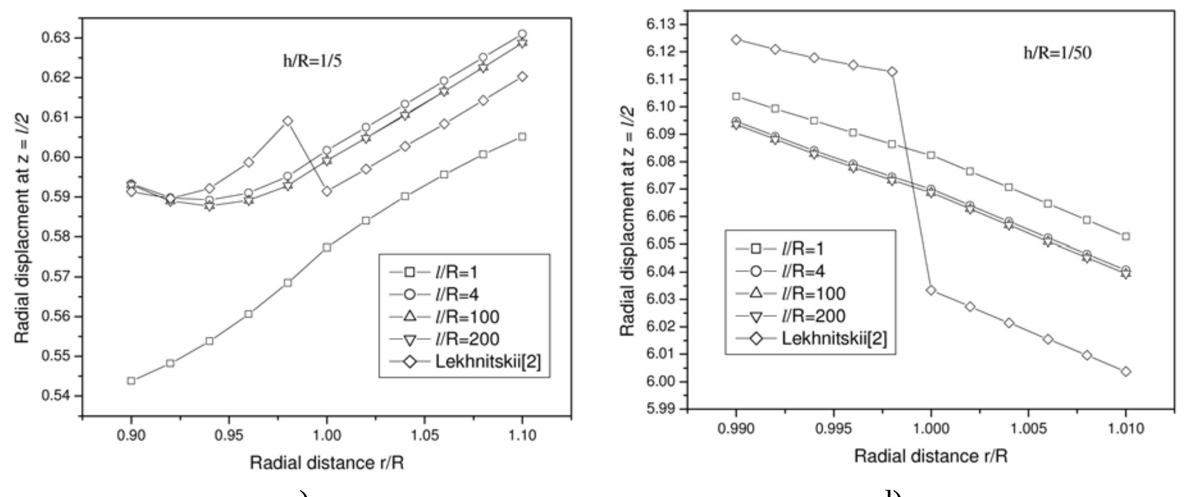


a) b)

Fig. 4. Distribution of shear stress $\overline{\tau_{rz}}$ and axial displacement $\overline{w}$ through thickness subjected to sinusoidal loading for orthotropic cylinder

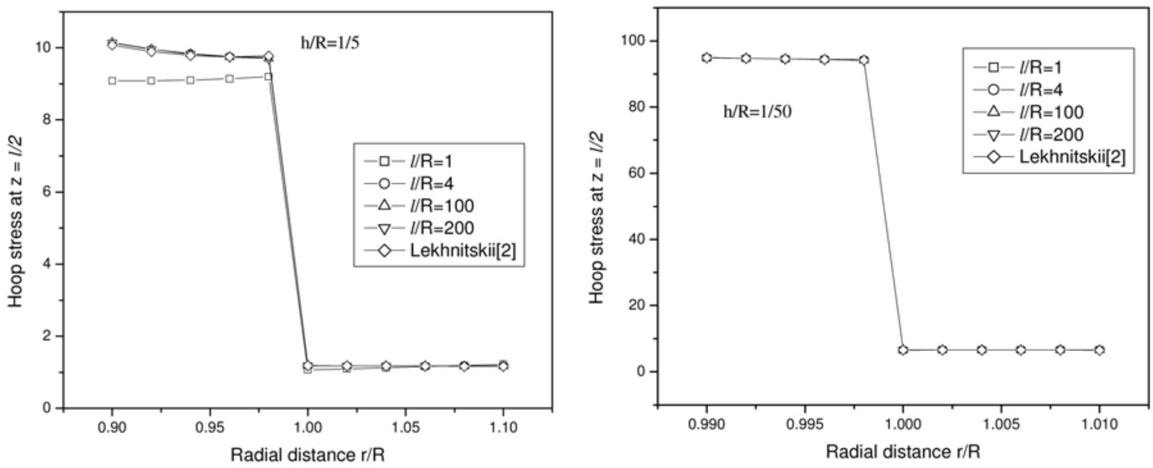


a) b)



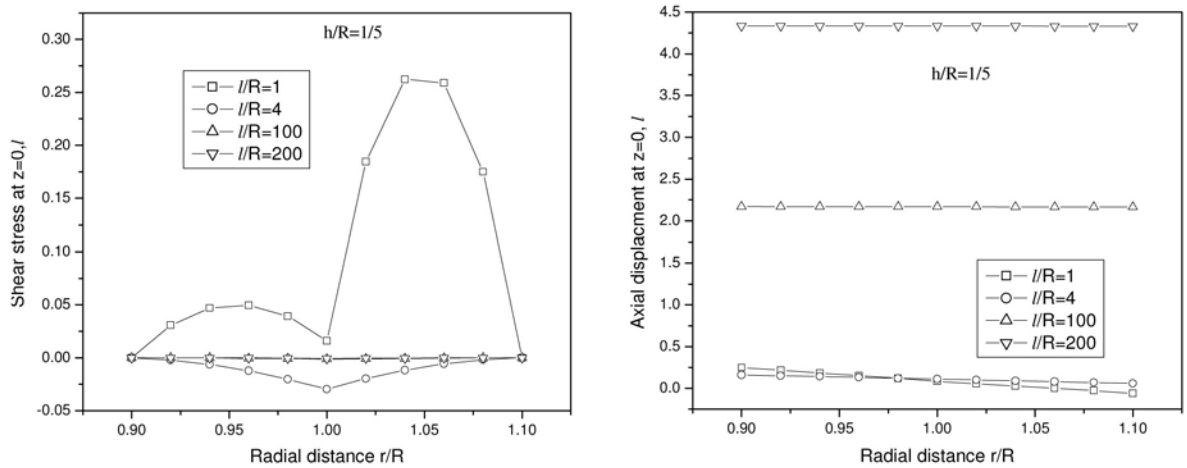
c) d)

Fig. 5. Distribution of radial stress $\overline{\sigma_r}$ and radial displacement $\overline{u}$ through thickness subjected to sinusoidal loading for layered ($0°/90°$) cylinder



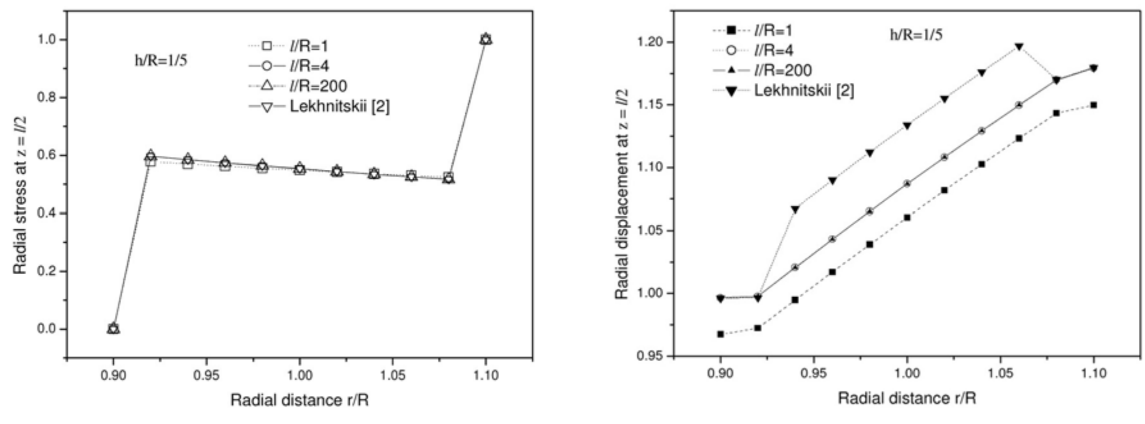
a) b)

Fig. 6. Distribution of hoop stress $\overline{\sigma_\theta}$ through thickness subjected to sinusoidal loading for layered $(0^\circ/90^\circ)$ cylinder



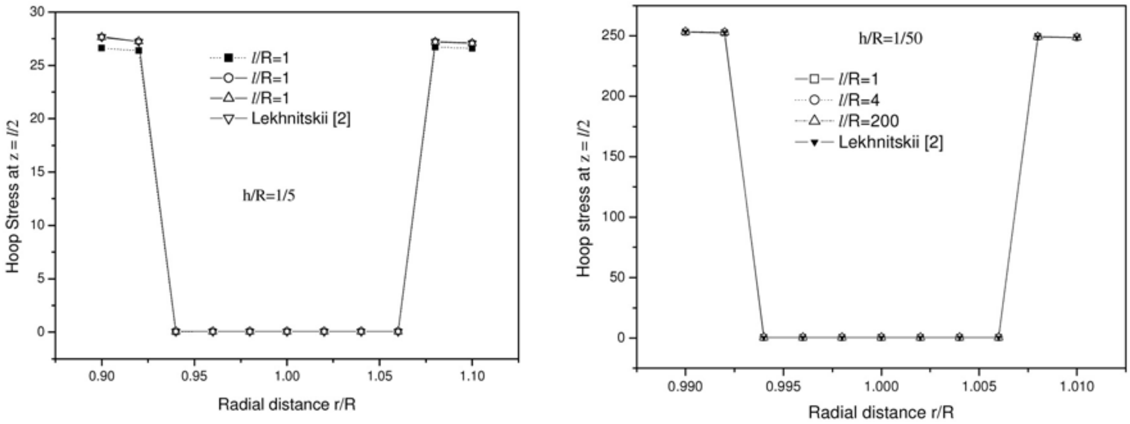
a) b)

Fig. 7. Distribution of shear stress $\overline{\tau_{rz}}$ and axial displacement $\overline{w}$ through thickness subjected to sinusoidal loading for layered $(0^\circ/90^\circ)$ cylinder



a) b)

Fig. 8. Distribution of radial stress $\overline{\sigma_r}$ and radial displacement $\overline{u}$ through thickness subjected to sinusoidal loading for sandwich cylinder



a) b)

Fig. 9. Distribution of hoop stress $\overline{\sigma_\theta}$ through thickness subjected to sinusoidal loading for sandwich cylinder

5. Conclusions

Numerical analysis of orthotropic, laminated fiber reinforced composite and sandwich cylinders under sinusoidal pressure loading is presented. Homogeneous and anisotropic media are considered under conditions of simply (diaphragm) supported cylinder. Exact analytical solutions are available only for infinitely long cylinders. The present results of cylinders of finite length are not only new but are also very accurate. Proposed numerical technique was found to be efficient, since 1) the derivation involves mixed variables, both displacements and stresses. 2) The continuity conditions between the layers are satisfied automatically while performing the numerical integration in radial coordinate.

Appendix: 1D Formulation for Orthotropic and Layered Cylinder

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0, \quad \varepsilon_r = \frac{\partial u}{\partial r} \quad \varepsilon_\theta = \frac{u}{r}, \tag{A1}$$

$$\begin{aligned} \sigma_r &= C_{11}\varepsilon_r + C_{12}\varepsilon_\theta, \qquad \sigma_r = C_{11}\frac{\mathrm{d}u}{\mathrm{d}r} + C_{12}\frac{u}{r}, \\ \sigma_\theta &= C_{12}\varepsilon_r + C_{22}\varepsilon_\theta, \qquad \sigma_\theta = C_{21}\frac{\mathrm{d}u}{\mathrm{d}r} + C_{22}\frac{u}{r}, \end{aligned}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\sigma_r}{C_{11}} - \frac{C_{12}}{C_{11}}\frac{u}{r}, \quad \frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = \frac{\sigma_r}{r}\left(\frac{C_{21}}{C_{11}} - 1\right) + \frac{u}{r^2}\left(C_{22} - \frac{C_{21}C_{12}}{C_{11}}\right), \tag{A2}$$

where

$$\nu_{r\theta} = \frac{\nu_{\theta r}}{E_\theta}E_r, \; C_{11} = \frac{E_r}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \; C_{12} = \frac{\upsilon_{r\theta}E_\theta}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \; C_{22} = \frac{E_\theta}{(1-\upsilon_{r\theta}\upsilon_{\theta r})}, \; C_{21} = C_{12}.$$

References

[1] Chandrashekhara, K., Kumar, B. S., Static analysis of a thick laminated circular cylindrical shell subjected to axisymmetric load. Composite Structures (23) (1993) 1–9.

[2] Gill, S., A Process for the step-by-Step integration of differential equations in an automatic digital computing machine. Proceedings of Cambridge Philosophical Society 1 (47) (1951) 96–108.

[3] Iyengar, K. T. S. R., Yogananda, C. V., Comparison of elasticity and shell theory solutions for long circular cylindrical shells, AIAA Journal 12 (4) (1966) 2 090–2 096.

[4] Kant, T., Numerical analysis of elastic plates with two opposite simply supported ends by segmentation method. Computers and Structures (14) (1981) 195–203.

[5] Kant, T., Numerical analysis of thick plates. Computer Methods in Applied Mechanics and Engineering (31) (1982) 1–18.

[6] Kant, T., Hinton, E., Mindlin plate analysis by segmentation method. ASCE Journal of Engineering Mechanics (109) (1983) 537–556.

[7] Kant, T., Ramesh, C. K., Numerical integration of linear boundary value problems in solid mechanics by segmentation method. International Journal of Numerical Methods in Engineering. (17) (1981) 1 233–1 256.

[8] Kant, T., Setlur, A. V., Computer analysis of clamped-clamped and clamped-supported cylindrical shells. Journal of the Aerospace Society of India 2 (25) (1973) 47–55.

[9] Klosner, J. M., Kempner, J., Comparison of elasticity and shell-theory solutions. AIAA Journal, 3 (1) (1963) 627–630.

[10] Klosner, J. M., Levine, H. S., Further comparison of elasticity and shell theory solutions. AIAA Journal 3 (4) (1966) 467–480.

[11] Kollar, L. P., Springer, G. S., Mechanics of Composite Structures. Cambridge University Press, New York, 2003.

[12] Lekhnitskii, S. G., Anisotropic Plates. Gordon and Breach Science, New York, 1968.

[13] Misovec, A. P., Kempner, J., Approximate elasticity solution for orthotropic cylinder under hydrostatic pressure and band loads. ASME Journal of Applied Mechanics 1 (37) (1970) 101–108.

[14] Pagano, N. J., Exact solutions for rectangular bidirectional composites and sandwich plates. Journal of Composite Materials (4) (1970) 21–34.

[15] Ramesh, C. K., Kant, T., Jadhav, V. B., Elastic analysis of cylindrical pressure vessels with various end closures. International Journal of Pressure Vessels and Piping (2) (1974) 143–154.

[16] Timoshenko, S., Goodier, J. N. Theory of Elasticity. McGraw-Hill, New York, 1951.

Nomenclature

r, θ, z	Cylindrical coordinates
u, v, w	Displacement components
σ_r, σ_θ, σ_z	Normal stress components on planes normal to r, θ, and z axes
τ_{zr}	Shearing stress components in cylindrical coordinates
ε_r, ε_θ, ε_z	Unit elongations (normal strains) in cylindrical coordinates
γ_{zr}	Shearing strain component in cylindrical coordinates
C_{ij}	Material constants for orthotropic material
ν	Poisson's ratio
r_i	Inner radius of the cylinder
r_0	Outer radius of the cylinder
l	Length of the cylinder
p	Uniform external pressure

$\overline{u}$, $\overline{w}$	Nondimensionalized displacement components
$\overline{\sigma_r}$, $\overline{\sigma_\theta}$, $\overline{\sigma_z}$	Nondimensionalized normal stress components
$\overline{\tau_{rz}}$	Nondimensionalized shearing stress in cylindrical coordinates
$\overline{r}$	Nondimensionalized radius
R	Mean radius $\frac{(r_0+r_i)}{2}$