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Steady Stokes flow of a non-Newtonian Reiner-Rivlin fluid streaming over an approximate liquid spheroid

B. R. Jaiswal[a,*]

[a] *Department of Applied Science, Symbiosis Institute of Technology, Lavale Hill Base Campus, Symbiosis International University (Deemed), Pune 412115, M.H., India*

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Abstract

The investigation is carried out to study steady Stokes axisymmetrical Reiner-Rivlin streaming flow over a fixed viscous droplet, and this droplet to be deformed sphere in shape. As boundary conditions, vanishing of radial velocities, continuity of tangential velocities and shear stresses at the droplet surface are used. The very common configuration of approximate sphere governed by polar equation $\tilde{r} = a[1 + \alpha_m \vartheta_m(\zeta)]$ has been considered for the study to $O(\alpha_m)$ describing the distortion. Based on the Stokes approximation, an analytical investigation is achieved in the orthogonal curve linear framework in an unbounded region of a Reiner-Rivlin fluid. In constraining cases, some earlier noted outcomes are obtained. Also, the yielded outcomes for the drag have been compared with solution existing in the literature. Further, the change for both force and pressure are evaluated showing deflection w.r.t. the parameters of interest and shown through table and graphs.
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1. Introduction

The transport process of a deformed solid or liquid particle immersed in an arbitrary fluid at very low Reynolds number (Re) is much of practical and fundamental interest as well. In the microscale domain of dynamic cells (motile), inertia is insignificant and the effect of viscous dissipation dominates the fluid forces on swimming bodies. To propel forwarding in such regime, many micro-organism deform their body shapes periodically by converting cell's chemical energy into mechanical work.

The classical theoretical solution of fluid flow problems is the viscous flow around a spherical body by Stokes [42] who himself derived the outcome of a spherical shaped body in an infinite fluid expanse. The solution given by Stokes accurately predicted drag force on a sphere. It is fact that, over the decades, creeping flow problems (Stokes flow) problems have been solved for the bodies which do not have the shape as sphere by plenty of researchers. But a few of the investigators reached to the solution for the non-spherical bodies like ellipsoid [21], hemisphere [23], and hemispherical cap [4] only. Sampson [36] was the first one who studied and carried out the creeping flow problem for the case of symmetrical flow for translation motion of a deformed body spherical in shape immersed in unbounded fluid phenomenon. Since the Sampson's work [36], several other researchers have achieved to formulate the flow phenomenon for such deformed spherical particles. Rybczynski [35] and Hadamard [12] researched desolately the fluid flow problems regarding translation movement of a fluid body in some other immiscible

*Corresponding author. Tel.: +91 822 589 26 02, e-mail: bharat.jaiswal@sitpune.edu.in.
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fluid and is given in their dissertation by Happel and Brenner [14]. Acrivos and Taylor [1] demonstrated the very common result of a steady viscous flow for an absolute particle of any shape in a vast liquid medium. The analytical approach to an arbitrary shape is quite difficult. Brenner [3] used perturbation technique to generalize the solution to such slightly distorted spheroids.

For very small Re, the inertia terms of the encompassing fluid proves to be unimportant contrasted with creeping effect and, commonly, receded. The Navier-Stokes (N-S) equations of fluid flow reduces to the Stokes equations to the 1st order of approximation and flow considered to become Stokes flow. There are many examples in our day-to-day life phenomenon in science like physical and biological, and in chemical engineering where deformed geometry plays a significant role in movement of the obstacle placed in fluid media. Indeed, the body with complex geometry in science and technology is less encountered in practice. In the later half of the nineteenth century, an intensive advanced development has happened ahead the application of creeping/Stokes flow over the body of arbitrary shape. Many different approaches have been developed and adopted to evaluate Stokes hydrodynamic drag force on an isolated axisymmetric body. Such an approach is sought by Datta and Srivastava [6] and the sequel of drag force on either of flow phenomenon (axial and transverse flow) were effectively calibrated for spheroids, prolates and spherical bodies. Using prolate spheroidal framework, Deo and Dutta [9] investigated the Stokes flow problems for a fluid prolate spheroid [9]. In addition to these bodies, some other shapes like egg-shaped, cycloidal, deformed sphere, and bodies of revolution under bearable error.

Rao and Rao [32] investigated the problem of slow motion of non-Newtonian micro-polar fluid streaming over a solid sphere, and evaluated the drag to be more on the spherical body as compared in the case of Newtonian fluid. Likewise, a similar kind of observation has been made by Aero et al. [2] and Stokes [43]. Jain [16] got the results for the Stokes flow motion of non-Newtonian fluid with invariable μ and μ_c by implementing the process of 'Synthetic division'. Sharma [38] contemplated the slow viscous movement of a non-Newtonian 2nd order fluid over a body of spherical in shape. Rathna [33] examined the creeping flow of Reiner-Rivlin liquid streaming over a rigid spherical body, and acquired a solution by utilizing the approximation given by Stokes. Later, the similar kind of technique utilized and implemented by Ramkissoon [25] to consider the fundamentally the same problem for a fluid sphere, and deduced that sphere encounters very much drag as compared Newtonian case.

The problem of creeping flow over an approximate deformed solid (stick) sphere is investigated by Happel and Brenner [13], and later this analysis has been, furthermore, carried out by Ramkissoon [24] for case of an approximate fluid sphere. In the analysis Ramkissoon has also justified the streaming over a spheroidal oblate particle as unique example. Also, it has additionally seen that for equivalent volume of both liquid sphere and fluid spheroid, liquid sphere encounters to a lesser extent obstruction. Ramkissoon and Majumadar [30] also observed the case of micro-polar fluid stream past a Newtonian liquid spheroid whose shape changes marginally from that of a sphere. Ramkissoon [29] examined the Stokes' case streaming over a non-Newtonian liquid spheroid. Stokes flow of a steady micro-polar liquid surpassing a deformed sphere has been examined [15], as well.

By employing blended slip-stick type conditions, an investigation was carried out by Palaniappan [22] and, later, solved the problem of viscous streams over a spheroid utilizing the very same conditions. In his study, the author got the conclusive mathematical demonstration for ψ, characterizing flow field to the $O(\varepsilon)$ in terms of Gegenbauer function and eccentricity of the spheroid, and analyzed that the slip-stick parameter expressively influences the drag force on the deformed sphere and, also observed a comparatively reduced drag on the approximate sphere as compared the drag on an oblate spheroid by Ramkissoon. Using the semi-separable technique, Dassios et al. [5] got the solution of axisymmetric creeping flow problem satisfying Stokes' equation in spheroidal coordinates and carried out an analysis over the problem of viscous flow streaming over spheroidal cell encompassing prolate spheroid. Recently, [20] examined the Stokes flow problem of Newtonian fluid over a micropolar liquid spheroid. In the right above discussed problems, the corresponding authors found that an increment in spin parameter results in a decreament to the drag force to micropolar fluid spheroid. The Stokes flow problem caused by a approximately spherical droplet in microstretch fluid was analytically explained by [39] and researched that the drag coefficients, all in all, are increasing functions of the micro-polarity, and decreasing with spin parameter.

Recently, an analysis was conducted to study viscous flow problem streaming over a deformed porous sphere implanted in some other permeable medium [11]. Following the numerous applications in different branches of engineering and science, the role of deformed particles in an assemblage has increased to a greater extent. Investigations regarding the symmetrical viscous flow past and through an assemblage of deformed porous spheroidal particles employing Happel boundary condition [7] and Kuwabara boundary condition [10]; using particle-in-cell technique [45, 46] and also influence of magnetic field on the Stokes flow problems [44] have been carried out.

Steady Stokes' flow about an approximate sphere was examined [40]. Srivastava et al. [41] discussed an analytical methodology for complete N-S equations by utilizing Oseen's relations for steady flow problems streaming over a spheroidal body. In all the literature mentioned above, the investigation was done only for flow assuming no-slip condition on the liquid-solid interface. The authors in [26, 27] studied the slip flow problems of Stokes' flow about a deformed sphere and viscoelastic fluid flow over a spheroid, respectively. Later, [28] investigated the similar type of polar flow problem streaming over a spheroid. The problem of slow viscous movement of a Reiner-Rivlin fluid (RRF) sphere immersed in an envelopment full of Newtonian fluid was envisioned and solved by [31]. Using prolate spheroidal coordinates, Deo and Dutta [8] investigated the Stokes flow problems for rigid prolate spheroidal with slip at the spheroidal surface. As of late, [18, 19] separately visioned the research on Reiner-Rivlin approximate fluid spheroid in confined and infinite expanse of fluids. Most recently, [17] examined Reiner-Rivlin streaming flow over a fluid spheroid, and assessed drag force experienced. The inspirations of these explorations, findings and availability of literature concerning deformed bodies(fluid and solid) enlivened to study the ongoing problem.

The literature suggests that no authors have considered previously deformed Newtonian fluid spherical body immersed in an unbounded Reiner-Rivlin fluid media. There are instances in literature where the study is greatly confined to the cases of viscous and micropolar fluids only. In the present brief note, to bridge the gap I have observed while going through the literature, I have considered the case of Reiner-Rivlin fluid and further extended the previous work by [17] to the fluid spheroid case and demonstrated how the results of Ramkissoon [25] can be employed to govern the Reiner-Rivlin fluid flow over a fluid spheroid analytically. The analytical expressions, to the $O(\varepsilon)$ of deformation parameter, for the stream function and other physical quantities are derived in closed forms as power series approximation. As a specific case, an oblate spheroidal fluid particle is taken for verification and then some distinctive and well-known outcomes including the modified expression for drag formula and pressure are derived for [25].

2. Formulation of the Problem

Consider the steady Stokes flow of an incompressible non-Newtonian Reiner-Rivlin fluid streaming over an approximate liquid spheroid. As the configuration of flow phenomenon delineated in Fig. 1, a system of spherical polar coordinates $(\tilde{r}, \theta, \phi)$ is taken with $-ve$ in z-axis as direction of the flow. The Reiner-Rivlin fluid is characterized by its rheological behavior in the form of constitutive equation (stress-strain relationship [34,37]) governed by (1) over a Newtonian fluid spheroid. Let the radius of the fluid spheroid be $\tilde{r} = a[1 + \chi(\theta)]$, where $\chi(\theta) = \alpha_m \vartheta_m(\zeta)$ with $\zeta = \cos\theta$, and the fluid spheroid held stationary in an unbounded flowing Reiner-Rivlin fluid with uniform velocity U away from obstacle. The flow fields are to be determined in the absence of body forces assuming the motion to be axially symmetric and fluids considered are immiscible in nature. The fluid viscosities are taken as (μ_e, μ_c) and μ_i for external and internal fluids, respectively. The parameters and constraints associated to the exterior and interior of the fluid spheroid to be denoted by the subscript/superscript 'e' and 'i' over an entity, respectively.

For an isotropic non-Newtonian Reiner-Rivlin fluid, the constitutive rheological equations for which the stress tensor $\tilde{\tau}_{ij}$ is related to the rate of strain tensor $\tilde{e}_{ij}$ as

$$\tilde{\tau}_{ij} = -\tilde{p}\delta_{ij} + \mu_e\,\tilde{e}_{ij} + \mu_c\,\tilde{e}_{ik}\,\tilde{e}_{kj}, \tag{1}$$

where $\tilde{p}$ is denoting the pressure, μ_e the coefficient of viscosity and μ_c is the coefficient of cross-viscosity, and

$$2\tilde{e}_{ij} = \tilde{v}_{i,j} + \tilde{v}_{j,i}, \tag{2}$$

here $\tilde{v}_i$ denotes i^{th} component of the fluid velocity.

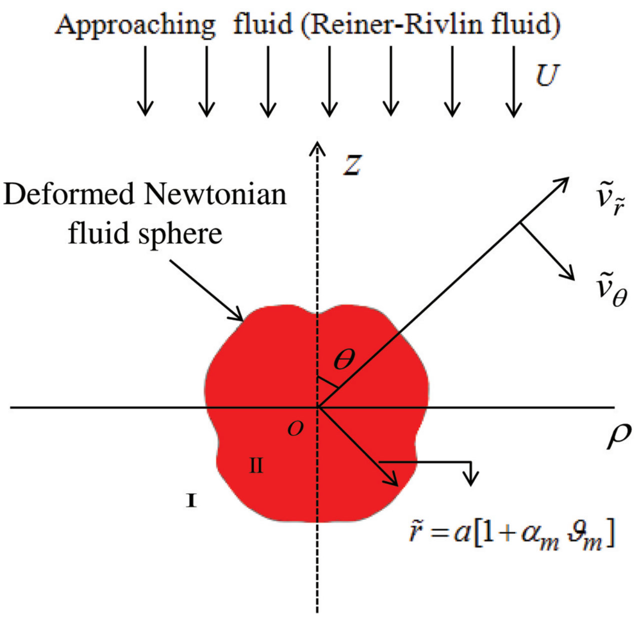

Fig. 1. Geometric delineation of flow domain and co-ordinate system

For the steady flow, the equations of continuity and momentum are as follows:

$$\tilde{v}_{k,k} = 0, \tag{3}$$

$$\tilde{\tau}_{ij,j} = \rho\,\tilde{v}_k\,\tilde{v}_{i,k}. \tag{4}$$

It is natural to describe the flow phenomenon in spherical polar coordinates $(\tilde{r}, \theta, \phi)$. In view of translational axis-symmetry, $\frac{\partial}{\partial\phi} \equiv 0$, we can express the velocity $\tilde{\mathbf{V}}$ as

$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}(\tilde{r}, \theta) = \big(\tilde{v}_{\tilde{r}}(\tilde{r}, \theta), \tilde{v}_\theta(\tilde{r}, \theta), 0\big), \tag{5}$$

where $\tilde{v}_{\tilde{r}}$ and $\tilde{v}_\theta$ denote the normal and tangential velocities to the surface of spheroid, respectively. Taking into consideration the condition of incompressibility (3) and introducing $\tilde{\psi}(\tilde{r}, \theta)$ by the subsequent relations

$$\tilde{v}_{\tilde{r}} = \frac{-1}{\tilde{r}^2 \sin\theta} \frac{\partial\tilde{\psi}}{\partial\theta}, \qquad \tilde{v}_\theta = \frac{1}{\tilde{r}\sin\theta} \frac{\partial\tilde{\psi}}{\partial\tilde{r}}, \qquad \tilde{v}_\phi = 0, \tag{6}$$

the following dimensional substitutionary parameters are used to make the quantities dimensionless

$$\tilde{r} = a\,r, \qquad \tilde{v}_{\tilde{r}} = U\,v_r, \qquad \tilde{v}_\theta = U\,v_\theta, \qquad \tilde{\tau}_{ij} = \mu\frac{U}{a}\tau_{ij},$$
$$\tilde{e}_{ij} = \frac{U}{a}e_{ij}, \qquad \tilde{p} = \mu\frac{U}{a}\,p, \qquad \tilde{\psi} = U\,a^2\psi,\ \lambda = \frac{\mu_e}{\mu_i},$$

where U is the fluid velocity far away from the obstacle, whereas a represents the radius of the sphere in exact form $\tilde{r} = a$ and $r, v_r, q_\theta, e_{ij}, \tau_{ij}, p, \psi, \ \ldots$ are non-dimensional parameters. The shear-stress components in non-dimensional forms reduce to

$$\begin{aligned}
\tau_{rr} &= -p + e_{rr} + S\left(e_{rr}^2 + e_{r\theta}^2\right), \\
\tau_{\theta\theta} &= -p + e_{\theta\theta} + S\left(e_{\theta\theta}^2 + e_{r\theta}^2\right), \\
\tau_{\phi\phi} &= -p + e_{\phi\phi} + S\,e_{\phi\phi}^2, \\
\tau_{r\theta} &= e_{r\theta} - S\,e_{r\theta}\,e_{\phi\phi},
\end{aligned} \tag{7}$$

where $S = \frac{\mu_c U}{\mu_e a}$. The momentum equations and continuity equation reduce to the following component forms under the Stokes approximation

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r}(2t_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \tau_{r\theta}\cot\theta) = 0, \tag{8}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{3}{r}\tau_{r\theta} + \frac{1}{r}(\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta = 0, \tag{9}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\,\sin\theta) = 0. \tag{10}$$

Expanding the parameters $v_r, v_\theta, e_{ij}, \tau_{ij}, p, \psi, \ \ldots$ in terms of power series of S as follows:

$$\begin{aligned}
\psi^{(e)} &= \psi_0 + \psi_1 S + \psi_2 S^2 + \ldots, \\
p^{(e)} &= p_0 + p_1 S + p_2 S^2 + \ldots, \ \text{etc.}
\end{aligned} \tag{11}$$

The subscript in (11) symbolize the 0^{th}, 1^{st}, 2^{nd} order approximations of the related quantities, and retaining the terms in the power series expansion of S up to 2^{nd} order only. Therefore, the stream functions ψ_0, ψ_1 and ψ_2 satisfy the following differential equations [25, 33]:

$$\begin{aligned}
&E^4\psi_0 = 0, \qquad E^4\psi_1 = 54\left(\frac{1}{r^5} - \frac{2}{r^7}\right)\vartheta_3(\zeta), \\
&E^4\psi_2 = \left(\frac{8}{5}S^2 h_2(r) + 2S^2 h_1(r)\right)\vartheta_2(\zeta) - \frac{8}{5}S^2 h_2(r)\vartheta_4(\zeta),
\end{aligned} \tag{12}$$

where

$$h_1(r) = \frac{27}{2r^6}\left(-7 + \frac{20}{r^5} - \frac{21}{r^3} - \frac{22}{r^2} + \frac{31}{r}\right), \quad h_2(r) = \frac{27}{4r^6}\left(16 - \frac{39}{r^5} + \frac{33}{r^3} + \frac{60}{r^2} - \frac{72}{r}\right).$$

The differential equations given by (12) yield the following solutions for stream function corresponding 0^{th}, 1^{st} and 2^{nd} order of approximations as

$$\begin{aligned} \psi_0 &= \left(r^2 + \frac{1}{2r} - \frac{3}{2}r\right)\vartheta_2(\zeta), \quad \psi_1 = \frac{3}{4}\left(1 - \frac{1}{r}\right)^3 \vartheta_3(\zeta), \\ \psi_2 &= \left[\frac{8}{5}S^2 f(r) + 2S^2 g(r)\right]\vartheta_2(\zeta) - \frac{8}{5}S^2 f(r)\vartheta_4(\zeta), \end{aligned} \tag{13}$$

where

$$\begin{aligned} f(r) &= \frac{9}{2\,464}\left[\frac{46}{r} - \frac{616}{r^2} + \frac{52}{r^3} + \frac{462}{r^4} + \frac{77}{r^5} - \frac{21}{r^7} + \frac{1\,056}{r^3}\log(r)\right], \\ g(r) &= \frac{1}{30\,800}\left[132r - \frac{3\,469}{r} + \frac{51\,975}{r^2} - \frac{1\,413}{r^3} - \frac{40\,425}{r^4} - \frac{9\,075}{r^5} + \frac{2\,275}{r^7} - \frac{95\,040}{r^3}\log(r)\right] \end{aligned}$$

and Stokes operator E^2 has the following form

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{14}$$

By Ramkissoon [25], we can take $\psi^{(e)}$ as

$$\psi^{(e)} = \psi_0 + \psi_1 S + \psi_2 S^2 + \sum_{n=2}^{\infty}\left(C_n r^{-n+1} + D_n r^{-n+3}\right)\vartheta_n(\zeta). \tag{15}$$

Simplifying (15) by utilizing (13) yields the external flow field as

$$\begin{aligned} \psi^{(e)} = &\left[r^2 - \frac{3}{2}r + \frac{1}{2r} + \frac{1}{r}c_2 + rd_2 + \frac{8}{5}S^2 f(r) + 2S^2 g(r)\right]\vartheta_2(\zeta) + \\ &\left[\frac{3}{4}S\left(1 - \frac{1}{r}\right)^3 + \frac{1}{r^2}c_3 + d_3\right]\vartheta_3(\zeta) + \left[\frac{1}{r^3}c_4 + \frac{1}{r}d_4 - \frac{8}{5}S^2 f(r)\right]\vartheta_4(\zeta) + \\ &\sum_{n=5}^{\infty}\left(C_n r^{-n+1} + D_n r^{-n+3}\right)\vartheta_n(\zeta). \end{aligned} \tag{16}$$

While for the Newtonian fluid in the inner region, the steady momentum equation for incompressible fluid with non-linear or inertia term neglected is given as

$$\nabla p = \mu_i \nabla^2 \mathbf{V}^{(\mathbf{i})}. \tag{17}$$

Taking curl of (17) and substituting (6) in non-dimensional form, the following Stokes equation is obtained:

$$E^4 \psi^{(i)} = 0. \tag{18}$$

The general solution to (18) for internal flows within the liquid spheroid is given as [13]

$$\psi^{(i)} = \left(a_2 r^2 + b_2 r^4\right)\vartheta_2(\zeta) + \sum_{n=3}^{\infty}\left(A_n r^n + B_n r^{n+2}\right)\vartheta_n(\zeta), \tag{19}$$

where $\zeta = \cos\theta$ and $\vartheta_n(\zeta)$ is the Gegenbauer function of 1st kind of order n and degree $-1/2$ connected to the Legendre polynomials $P_n(\zeta)$ by

$$\vartheta_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad n \geq 2, \qquad P_n(\zeta) = \frac{1}{2^n n!}\frac{\partial^n}{\partial\zeta^n}\left(\zeta^2 - 1\right)^n$$

with

$$\begin{aligned}
&P_0(\zeta) = 1, \quad P_1(\zeta) = \zeta, \quad P_2(\zeta) = \frac{1}{2}\left(-1 + 3\zeta^2\right), \quad P_3(\zeta) = \frac{1}{2}\zeta\left(-3 + 5\zeta^2\right),\\
&P_4(\zeta) = \frac{1}{8}\left(3 - 30\zeta^2 + 35\zeta^4\right), \quad P_5(\zeta) = \frac{1}{8}\zeta\left(15 - 70\zeta^2 + 63\zeta^4\right),\\
&P_6(\zeta) = \frac{1}{16}\left(-5 + 105\zeta^2 - 315\zeta^4 + 231\zeta^6\right).
\end{aligned}$$

Also, these relations have the accompanying special identities framed by Happel and Brenner [13] relevant to our work

$$\begin{aligned}
\vartheta_m(\zeta)\,\vartheta_2(\zeta) &= \lambda_{m-2}\,\vartheta_{m-2}(\zeta) + \lambda_m\,\vartheta_m(\zeta) + \lambda_{m+2}\,\vartheta_{m+2}(\zeta),\\
\vartheta_m(\zeta)\,\vartheta_4(\zeta) &= \phi_{m-4}\,\vartheta_{m-4}(\zeta) + \phi_{m-2}\,\vartheta_{m-2}(\zeta) + \phi_m\,\vartheta_m(\zeta) +\\
&\quad \phi_{m+2}\,\vartheta_{m+2}(\zeta) + \phi_{m+4}\,\vartheta_{m+4}(\zeta),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\lambda_{k+2} &= -\frac{(1+k)(2+k)}{2(-1+2k)(1+2k)}, \quad \lambda_k = \frac{k(-1+k)}{(1+2k)(-3+2k)},\\
\lambda_{k-2} &= -\frac{(-3+k)(-2+k)}{2(-3+2k)(-1+2k)},\\
\phi_{k+4} &= -\frac{5(k+1)(2+k)(3+k)(k+4)}{8(5+2k)(-1+2k)(3+2k)(1+2k)},\\
\phi_{k+2} &= \frac{k(1+k)(2+k)\left(16k^3 + 48k^2 + 44k + 12\right)}{8(-3+2k)(-1+2k)(3+2k)(1+2k)^2(5+2k)},\\
\phi_k &= \frac{k^2\left(8k^5 - 28k^4 + 30k^3 - 5k^2 - 8k + 3\right)}{4(-3+2k)^2(-1+2k)(3+2k)(1+2k)^2(-5+2k)},\\
\phi_{k-2} &= \frac{(k-2)(k-3)\left(4k^4 - 28k^3 + 71k^2 - 77k + 30\right)}{2(-7+2k)(-1+2k)(-3+2k)^2(1+2k)(-5+2k)},\\
\phi_{k-4} &= -\frac{5(k-3)(-2+k)(-5+k)(k-4)}{8(-7+2k)(-5+2k)(-3+2k)(-1+2k)}.
\end{aligned} \tag{21}$$

Here, the coefficients contributing to the flow streaming a fluid sphere [25] are c_2, d_2, c_4, d_4, a_2, b_2, a_4, b_4 only, and consequently, all other coefficients in (16) and (19) are of the first order in α_k. In this manner, with the exception of where the coefficients c_2, d_2, c_4, d_4, a_2, b_2, a_4, b_4 appeared, we may take the surface to be $\tilde{r} = a$ or $r = 1$ rather than its exact form (22).

Let $\tilde{r} = a[1 + \chi(\theta)]$ be the equation of spheroidal surface S_d in polar form approximating the sphere $\tilde{r} = a$. The orthogonality of Gegenbauer polynomials, commonly, enables us to take $\chi(\theta)$ as the following form

$$\chi(\theta) = \sum_{k=2}^{\infty} \alpha_k \vartheta_k(\zeta).$$

Hence, we can take the surface S_d [30]

$$\tilde{r} = a[1 + \alpha_k \vartheta_k(\zeta)], \tag{22}$$

where α_k to be adequately minute so that the higher powers may be over looked, i.e.,

$$\left(\frac{\tilde{r}}{a}\right)^{\ell} \approx 1 + \ell\, \alpha_k\, \vartheta_k(\zeta),$$

where ℓ is $+ve$ or $-ve$.

3. Employment of boundary conditions to evaluate arbitrary unknown

The arbitrary constants encountering in (16) to be evaluated by employing boundary conditions over the spheroidal surface $r = 1 + \alpha_k \vartheta_k(\zeta)$. In terms of stream function, these boundary conditions reduce to

Impenetrability condition:
$$\psi^{(e)} = 0, \quad \psi^{(i)} = 0, \tag{23}$$

Continuity of tangential velocities:
$$\frac{\partial \psi^{(e)}}{\partial r} = \frac{\partial \psi^{(i)}}{\partial r}, \tag{24}$$

Continuity of tangential stresses:
$$\lambda \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi^{(e)}}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \psi^{(i)}}{\partial r}\right), \tag{25}$$

where $\lambda = \mu_e/\mu_i$. Application of the boundary conditions (23)–(25) yields the following system of linear equations

$$\begin{gathered}(c_2 + d_2)\vartheta_2(\zeta)(c_2 - d_2)\alpha_k\vartheta_k(\zeta)\vartheta_2(\zeta) + (d_3 + c_3)\vartheta_3(\zeta) + \\ (c_4 + d_4)\vartheta_4(\zeta) - (3c_4 + d_4)\alpha_k\vartheta_k(\zeta)\vartheta_4(\zeta) + \sum_{n=5}^{\infty}(D_n + C_n)\vartheta_n(\zeta) = 0,\end{gathered} \tag{26}$$

$$\begin{gathered}(a_2 + b_2)\vartheta_2(\zeta) + 2(a_2 + 2b_2)\alpha_k\vartheta_k(\zeta)\vartheta_2(\zeta) + (a_3 + b_3)\vartheta_3(\zeta) + \\ (a_4 + b_4)\vartheta_4(\zeta) + (4a_4 + 6b_4)\alpha_k\vartheta_k(\zeta)\vartheta_4(\zeta) + \sum_{n=5}^{\infty}(A_n + B_n)\vartheta_n(\zeta) = 0,\end{gathered} \tag{27}$$

$$\begin{gathered}-(2a_2 + 4b_2 + c_2 - d_2)\vartheta_2(\zeta) + \\ \left(3 + \frac{3S^2}{700} - 2a_2 - 12b_2 + 2c_2\right)\alpha_k\vartheta_2(\zeta)\vartheta_k(\zeta) - \\ (3a_3 + 5b_3 + 2c_3)\vartheta_3(\zeta) - (4a_4 + 6b_4 + 3c_4 + d_4)\vartheta_4(\zeta) +\end{gathered}$$

$$\frac{1}{770}(-9S^2 + 9\,240c_4 + 1\,540d_4)\vartheta_4(\zeta)\vartheta_k(\zeta)\alpha_k +$$
$$\sum_{n=5}^{\infty}[-nA_n - (2+n)B_n - (-3+n)D_n - (-1+n)C_n]\vartheta_n(\zeta) = 0, \tag{28}$$

$$\left[2a_2 - 4b_2 + \lambda\left(3 + \frac{3}{700}S^2 + 4c_2 - 2d_2\right)\right]\vartheta_2(\zeta) +$$
$$\left\{-4a_2 + \lambda\left[-20c_2 + 3\left(-5 + \frac{1}{70}S^2 + 2d_2\right)\right]\right\}\alpha_k\vartheta_2(\zeta)\vartheta_k(\zeta) -$$
$$10(b_3 - \lambda c_3)\vartheta_3(\zeta) - \left[4a_4 + 18b_4 + \lambda\left(\frac{9}{770}S^2 - 18c_4 - 4d_4\right)\right]\vartheta_4(\zeta) +$$
$$\lambda\left(\frac{9}{110}S^2 - 126c_4 - 20d_4\right)\alpha_k\vartheta_4(\zeta)\vartheta_k(\zeta) +$$
$$\sum_{n=5}^{\infty}\{-(-3+n)nA_n - (-2+n+n^2)B_n +$$
$$\lambda[(-3+n)nD_n + (-2+n+n^2)C_n]\}\vartheta_n(\zeta) = 0. \tag{29}$$

The leading terms, i.e., the coefficients ϑ_2, ϑ_3, ϑ_4 appearing in equations (26)–(29) must be vanished, i.e.,

$$\begin{aligned}
&c_2 + d_2 = 0, \quad c_3 + d_3 = 0, \quad c_4 + d_4 = 0, \quad a_2 + b_2 = 0, \quad a_3 + b_3 = 0,\\
&a_4 + b_4 = 0, \quad 2a_2 + 4b_2 + c_2 - d_2 = 0, \quad 3a_3 + 5b_3 + 2c_3 = 0,\\
&4a_4 + 6b_4 + 3c_4 + d_4 = 0, \quad 2a_2 - 4b_2 + \lambda\left(3 + \frac{3}{700}S^2 + 4c_2 - 2d_2\right) = 0,\\
&b_3 - \lambda c_3 = 0, \quad 4a_4 + 18b_4 + \lambda\left(\frac{9}{770}S^2 - 18c_4 - 4d_4\right) = 0.
\end{aligned} \tag{30}$$

Solving (30) gives

$$\begin{aligned}
&c_2 = -\frac{700\lambda + S^2\lambda}{1\,400(1+\lambda)}, \quad d_2 = \frac{700\lambda + S^2\lambda}{1\,400(1+\lambda)}, \quad c_4 = \frac{9S^2\lambda}{10\,780(1+\lambda)}, \quad d_4 = -\frac{9S^2\lambda}{10\,780(1+\lambda)},\\
&a_2 = -\frac{700\lambda + S^2\lambda}{1\,400(1+\lambda)}, \quad b_2 = \frac{700\lambda + S^2\lambda}{1\,400(1+\lambda)}, \quad a_4 = \frac{9S^2\lambda}{10\,780(1+\lambda)}, \quad b_4 = -\frac{9S^2\lambda}{10\,780(1+\lambda)},\\
&a_3 = 0, \quad b_3 = 0, \quad c_3 = 0, \quad d_3 = 0.
\end{aligned} \tag{31}$$

These are the same values of unknowns which were also noticed by Ramkissoon [25] for the case of fluid sphere. Therefore, equations (26)–(29) now reduce to

$$\chi_{11}\,\alpha_k\vartheta_k(\zeta)\vartheta_2(\zeta) + \chi_{13}\,\alpha_k\vartheta_k(\zeta)\vartheta_4(\zeta) - \sum_{n=5}^{\infty}(D_n + C_n)\vartheta_n(\zeta) = 0, \tag{32}$$

$$\chi_{21}\,\alpha_k\vartheta_k(\zeta)\vartheta_2(\zeta) + \chi_{23}\,\alpha_k\vartheta_k(\zeta)\vartheta_4(\zeta) + \sum_{n=5}^{\infty}(A_n + B_n)\vartheta_n(\zeta) = 0, \tag{33}$$

$$\chi_{31}\,\alpha_k\vartheta_2(\zeta)\vartheta_k(\zeta) + \chi_{33}\,\vartheta_4(\zeta)\vartheta_k(\zeta)\alpha_k +$$
$$\sum_{n=5}^{\infty}\{-nA_n - (2+n)B_n - (-3+n)D_n - (-1+n)C_n\}\vartheta_n(\zeta) = 0, \tag{34}$$

$$\chi_{41}\,\alpha_k\vartheta_2(\zeta)\vartheta_k(\zeta)+\chi_{43}\,\alpha_k\vartheta_4(\zeta)\vartheta_k(\zeta)+$$
$$\sum_{n=5}^{\infty}\left[-(-3+n)nA_n-\left(-2+n+n^2\right)B_n+\right.$$
$$\left.\lambda\left\{(-3+n)nD_n+\left(-2+n+n^2\right)C_n\right\}\right]\vartheta_n(\zeta)=0, \tag{35}$$

where

$$\chi_{11}=-\frac{700\lambda+S^2\lambda}{700(1+\lambda)},\quad \chi_{13}=\frac{9S^2\lambda}{5\,390(1+\lambda)},\quad \chi_{21}=\frac{700\lambda+S^2\lambda}{700(1+\lambda)},$$
$$\chi_{23}=-\frac{9S^2\lambda}{5\,390(1+\lambda)},\quad \chi_{31}=-\frac{3(700+S^2)(-1+\lambda)}{700(1+\lambda)},\quad \chi_{33}=-\frac{9S^2(7+2\lambda)}{5\,390(1+\lambda)}, \tag{36}$$
$$\chi_{41}=\frac{\lambda\left(-700(13+2\lambda)+S^2(32+43\lambda)\right)}{700(1+\lambda)},\quad \chi_{43}=-\frac{9S^2\lambda(-49+4\lambda)}{5\,390(1+\lambda)}.$$

Solving (32)–(35) with the aid of the relations (20), we see that the coefficients become identically zero for all n except $n=k-4,k-2,k,k+2,k+4$. And hence, the non-vanishing constants yield the following expressions

$$A_n=\frac{1}{2(-1+2n)(1+\lambda)}\alpha_k\Big[\lambda_n\{\left(3-4n+n^2\right)\lambda\chi_{11}-(2+n)(-1+n-\lambda+2n\lambda)\chi_{21}+$$
$$\lambda\chi_{31}-2n\lambda\chi_{31}-\chi_{41}\}+\phi_n\{\left(3-4n+n^2\right)\lambda\chi_{13}-(2+n)(-1+n-\lambda+2n\lambda)\times$$
$$\chi_{23}+\lambda\chi_{33}-2n\lambda\chi_{33}-\chi_{43}\}\Big], \tag{37}$$
$$B_n=-\frac{1}{2(-1+2n)(1+\lambda)}\alpha_k\Big[\lambda_n\{\left(3-4n+n^2\right)\lambda\chi_{11}+n(3-n+\lambda-2n\lambda)\chi_{21}+\lambda\chi_{31}-$$
$$-2n\lambda\chi_{31}-\chi_{41}\}+\phi_n\{\left(3-4n+n^2\right)\lambda\chi_{13}+n(3-n+\lambda-2n\lambda)\chi_{23}+\lambda\chi_{33}-$$
$$-2n\lambda\chi_{33}-\chi_{43}\}\Big], \tag{38}$$
$$C_n=-\frac{1}{2(-1+2n)(1+\lambda)}\alpha_k\Big[\lambda_n\{(-3+n)(-1+n(2+\lambda))\chi_{11}-n(2+n)\chi_{21}+\chi_{31}-$$
$$-2n\chi_{31}+\chi_{41}\}+\phi_n\{(-3+n)(-1+n(2+\lambda))\chi_{13}-n(2+n)\chi_{23}+\chi_{33}-$$
$$-2n\chi_{33}+\chi_{43}\}\Big], \tag{39}$$
$$D_n=\frac{1}{2(-1+2n)(1+\lambda)}\alpha_k\Big[\lambda_n\{(-1+n)(-1+2\lambda+n(2+\lambda))\chi_{11}-n(2+n)\chi_{21}+$$
$$\chi_{31}-2n\chi_{31}+\chi_{41}\}+\phi_n\{(-1+n)(-1+2\lambda+n(2+\lambda))\chi_{13}-n(2+n)\chi_{23}+$$
$$\chi_{33}-2n\chi_{33}+\chi_{43}\}\Big]. \tag{40}$$

Hence, we know all the coefficients now. Therefore, the flow fields characterizing the motions are now determined as

$$\psi^{(e)}=\left[r^2-\frac{3}{2}r+\frac{1}{2r}+\frac{1}{r}c_2+r\,d_2+\frac{8}{5}S^2f(r)+2S^2g(r)\right]\vartheta_2(\zeta)+$$
$$\left[\frac{3}{4}S\left(1-\frac{1}{r}\right)^3\right]\vartheta_3(\zeta)+\left[\frac{1}{r^3}c_4+\frac{1}{r}d_4-\frac{8}{5}S^2f(r)\right]\vartheta_4(\zeta)+$$

$$\left(C_{k-4}r^{-k+5}+D_{k-4}r^{-k+7}\right)\vartheta_{k-4}(\zeta)+\left(C_{k-2}r^{-k+3}+D_{k-2}r^{-k+5}\right)\vartheta_{k-2}(\zeta)+ \left(C_{k}r^{-k+1}+D_{k}r^{-k+3}\right)\vartheta_{k}(\zeta)+\left(C_{k+2}r^{-k-1}+D_{k+2}r^{-k+1}\right)\vartheta_{k+2}(\zeta)+ \left(C_{k+4}r^{-k-3}+D_{k+4}r^{-k-1}\right)\vartheta_{k+4}(\zeta) \tag{41}$$

and

$$\psi^{(i)}=\left(a_2r^2+b_2r^4\right)\vartheta_2(\zeta)+\left(a_4r^4+b_4r^6\right)\vartheta_4(\zeta)+\left(A_{k-4}r^{k-4}+B_{k-4}r^{k-2}\right)\vartheta_{k-4}(\zeta)+ \left(A_{k-2}r^{k-2}+B_{k-2}r^{k}\right)\vartheta_{k-2}(\zeta)+\left(A_{k}r^{k}+B_{k}r^{k+2}\right)\vartheta_{k}(\zeta)+ \left(A_{k+2}r^{k+2}+B_{k+2}r^{k+4}\right)\vartheta_{k+2}(\zeta)+\left(A_{k+4}r^{k+4}+B_{k+4}r^{k+6}\right)\vartheta_{k+4}(\zeta), \tag{42}$$

where all A_n, B_n, C_n and D_n were determined previously. And consequently yields the solution for the proposed problem.

4. Application to an oblate spheroid

We now take an example of oblate spheroidal body to validate the analysis. The flow is assumed to be in the direction to its symmetrical axis (see Fig. 2). Our main objective is to evaluate the force on it. The equation in Cartesian co-ordinates (x, y, z) of spheroidal particle can be taken as

$$\frac{x^2+y^2}{b^2}+\frac{z^2}{b^2(1-\varepsilon)^2}=1, \tag{43}$$

where b is the equatorial radius, ε is a deformation parameter such that its powers higher than unity may be disregarded and $\varepsilon > 0$ for the considered geometry. Thus, equation (43) in polar co-ordinates (ρ, ζ) yields the following

$$\tilde{r}=a[1+2\varepsilon\,\vartheta_2(\zeta)], \tag{44}$$

where $a = b(1-\varepsilon)$. To utilize the outcomes, inserting $k = 2$, $\alpha_k = 2\varepsilon$ in the previous section. Therefore, the non-vanishing coefficients are gotten just when $n = k, k+2, k+4$. In this manner, stream functions characterizing outer and inner flow fields yield the accompanying expressions:

$$\psi^{(e)}=\{r^2-\frac{3}{2}r+\frac{1}{2r}+\frac{1}{r}c_2+r\,d_2+\frac{8}{5}S^2f(r)+2S^2g(r)+C_2r^{-1}+D_2r\}\vartheta_2(\zeta)+ \frac{3}{4}S\left(1-\frac{1}{r}\right)^3\vartheta_3(\zeta)+\{\frac{1}{r^3}c_4+\frac{1}{r}d_4-\frac{8}{5}S^2f(r)+C_4r^{-3}+D_4r^{-1}\}\vartheta_4(\zeta)+ \left(C_6r^{-5}+D_6r^{-3}\right)\vartheta_6(\zeta) \tag{45}$$

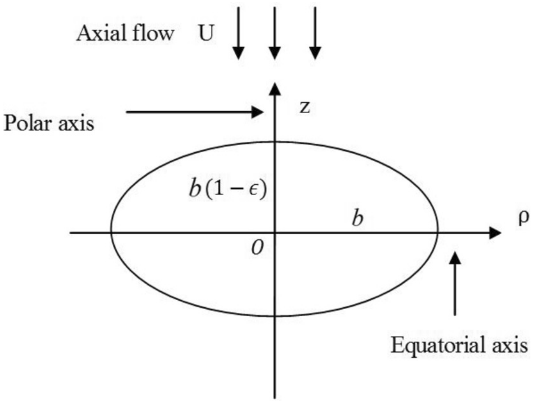

Fig. 2. Geometric sketch and co-ordinate system of an oblate spheroidal particle in meridian plane

and

$$\psi^{(i)} = \left(a_2 r^2 + b_2 r^4 + A_2 r^2 + B_2 r^4\right)\vartheta_2(\zeta) + \left(A_4 r^4 + B_4 r^6\right)\vartheta_4(\zeta) + \left(A_6 r^6 + B_6 r^8\right)\vartheta_6(\zeta). \tag{46}$$

5. Assessment of drag force on oblate fluid spheroid

For flow phenomenon, the most important physical highlights is the drag experienced by the obstacle immersed in fluid. To assess this, we utilize a straightforward elegant formula [23], and for the present case, this reduces to

$$F_z = 8\pi\mu_e \lim_{\tilde{r}\to\infty} \frac{\tilde{\psi}_e - \tilde{\psi}_\infty}{2\tilde{r}\,\vartheta_2(\zeta)}, \tag{47}$$

where $\tilde{\psi}_\infty$ represents a flow field far away from the body and is equivalent to

$$\tilde{\psi}_\infty = \tilde{r}^2 U \vartheta_2(\zeta) \tag{48}$$

and

$$\begin{aligned}\tilde{\psi}_e = \mathrm{Ub}^2 \Bigg[& \left\{ \frac{\tilde{r}^2}{b^2} + (1-\varepsilon)\left(-\frac{3}{2} + d_2 + D_2\right)\left(\frac{\tilde{r}}{b}\right) + (1-3\varepsilon)\left(\frac{1}{2} + c_2 + C_2\right)\left(\frac{b}{\tilde{r}}\right) + \right. \\ & \left. 2\varepsilon S^2 \left(\frac{4}{5} f'\left(\frac{\tilde{r}}{b}\right) + g'\left(\frac{\tilde{r}}{b}\right)\right)\left(\frac{\tilde{r}}{b}\right) + 2(1-2\varepsilon)S^2\left(\frac{4}{5} f\left(\frac{\tilde{r}}{b}\right) + g\left(\frac{\tilde{r}}{b}\right)\right)\right\}\vartheta_2(\zeta) + \\ & \frac{3}{4}S\left\{\left(1-\frac{b}{\tilde{r}}\right)^3 + \varepsilon\left(5\frac{b}{\tilde{r}} - 2\right)\left(1 - \frac{b}{\tilde{r}}\right)^2\right\}\vartheta_3(\zeta) + \left\{(1-5\varepsilon)(c_4 + C_4)\left(\frac{b^3}{\tilde{r}^3}\right) + \right. \\ & \left. (1-3\varepsilon)(d_4 + D_4)\left(\frac{b}{\tilde{r}}\right) - (1-2\varepsilon)\frac{8}{5}S^2 f\left(\frac{\tilde{r}}{b}\right) - \frac{8}{5}S^2\varepsilon f'\left(\frac{\tilde{r}}{b}\right)\left(\frac{\tilde{r}}{b}\right)\right\}\vartheta_4(\zeta) + \\ & \left\{(1-7\varepsilon)\left(\frac{b^5}{\tilde{r}^5}\right)C_6 + (1-5\varepsilon)\left(\frac{b^3}{\tilde{r}^3}\right)D_6\right\}\vartheta_6(\zeta)\Bigg].\end{aligned} \tag{49}$$

Substitution of (48) and (49) into (47) yields

$$F_z = -\frac{2}{175} b\pi U(-1+\varepsilon)\left(-525 + 3S^2 + 350 d_2 + 350 D_2\right)\mu_e, \tag{50}$$

where

$$\begin{aligned} d_2 &= \frac{700\lambda + S^2\lambda}{1\,400(1+\lambda)}, \\ D_2 &= \frac{\varepsilon\left\{-377\,300\,(3 + 5\lambda + 2\lambda^2) + S^2\left(-1\,932 + 4\,520\lambda + 7\,007\lambda^2\right)\right\}}{943\,250(1+\lambda)^2}. \end{aligned} \tag{51}$$

Hence, from (50) which yields the expression for drag experienced by Newtonian fluid spheroid immersed in a Reiner-Rivlin fluid, the accompanying well familiar cases can be extracted immediately:

1. When $\varepsilon = 0$, Ramkissoon's formula [25] in corrected form for Reiner-Rivlin flow past a fluid sphere

$$F_z = -6b\pi\mu_e U\left\{\frac{(1+\frac{2}{3}\lambda)}{1+\lambda} - S^2\left(\frac{12}{700(1+\lambda)} + \frac{13}{700(1+\lambda)}\lambda\right)\right\}. \tag{52}$$

2. When $\lambda = 0$, new result [17] for Reiner-Rivlin liquid over a rigid spheroid

$$F_z = -6b\pi\mu_e U \left\{ \left(1 - \frac{\varepsilon}{5}\right) + S^2(-0.005\,7 + 0.007\,1\varepsilon) \right\}. \tag{53}$$

3. When $S = 0,$ famous Ramkissoon's formula [24] for a Newtonian fluid past a Newtonian fluid spheroid

$$F_z = -6b\pi U \left(1 - \frac{\varepsilon}{5}\right) \frac{\left(1 + \frac{2}{3}\lambda\right) \mu_e}{(1 + \lambda)}. \tag{54}$$

4. When $\lambda = 0, \varepsilon = 0$, Rathna's formula [33] in corrected form for Reiner-Rivlin liquid past a rigid sphere

$$F_z = -6b\pi\mu_e U \left(1 - 0.005\,7\, S^2\right). \tag{55}$$

5. When $S = 0$, $\lambda = 0$ renowned Stokes' formula [42] for a viscous fluid past a rigid spheroid

$$F_z = -6b\pi\mu_e U \left(1 - \frac{\varepsilon}{5}\right). \tag{56}$$

Figs. 3–7 show the effect of cross viscosity S, relative viscosity λ as well as the deformation parameter ε on hydrodynamic drag and pressure, and the corresponding numerical values are listed in Table 1. One can easily determine the impact of S, λ and ε on F_z and pressure.

Table 1. Comparisons of $F_z/(-6a\pi\mu_e U)$ calculated for solid and liquid spheroid for $S = 0$ and $S = 0.5$

$F_z/(-6a\pi\mu_e U) \longrightarrow$				
ε	$\lambda = 0$		$\lambda = 0.5$	
$\downarrow$	$S = 0$	$S = 0.5$	$S = 0$	$S = 0.5$
0.00	1.000	0.998 571 429	0.888 888 889	0.887 420 635
0.01	0.998	0.996 589 128	0.887 111 111	0.885 655 906
0.20	0.960	0.958 925 417	0.853 333 333	0.852 126 065
0.40	0.920	0.919 279 406	0.817 777 778	0.816 831 496
0.60	0.880	0.879 633 395	0.782 222 222	0.781 536 926
0.80	0.840	0.839 987 384	0.746 666 667	0.746 242 357
0.99	0.802	0.802 323 673	0.712 888 889	0.712 712 515
1.00	0.800	0.800 341 373	0.711 111 111	0.710 947 787

The changes on $F_z/(-6a\pi\mu_e U)$ for spheroid w.r.t. ε are exhibited in Fig. 3 for numerous values of S. From this figure, it is seen that the drag $F_z/(-6a\pi\mu_e U)$ on a liquid sphere is more than such force acknowledged by an oblate fluid spheroid. For lesser deformation, the force is more. Also, when the disfigurement expands, the force continues to diminishing with identical intensity of ε at later stages. We likewise witness that force on spheroid for viscous case is less as compared to the Reiner-Rivlin fluid case.

Fig. 4 shows the dependence of dimensionless hydrodynamic drag force on deformation parameter ε at $S = 0.75$ and for different values of relative viscosity λ. The drag force decreases with deformity ε. This decrease in drag force is sharp for higher deformation.

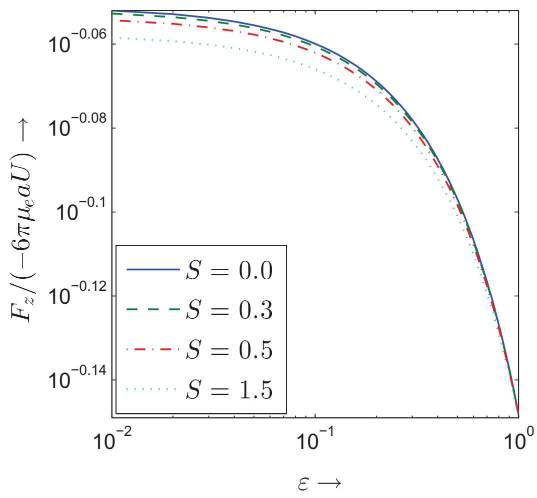

Fig. 3. The drag force on viscous droplet for different parameter S and $\lambda = 0.5$

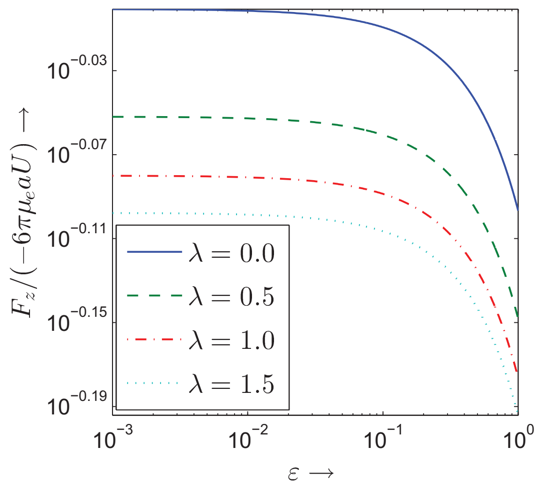

Fig. 4. The drag force on viscous droplet for different parameter λ and $S = 0.5$

The variation of drag force w.r.t. relative viscosity λ is depicted in Fig. 5 for various values of cross-viscous parameter S. The graphs show that initially the drag decreases slowly and then gets reducing sharply at subsequent stages. We also noticed a constant decrease in drag for isolated values of S.

The influence of S on F_z is exhibited in Fig. 6. These graphs show that F_z diminishes as ε increases. Also, it is observed and analyzed that F_z, for smaller S, turns out to be practically consistent for an estimation of ε. Yet, for bigger estimations of cross-viscosity (greater than 0.9), the drag diminishes quickly for some random estimation of ε.

The impact of deformation ε on the non-dimensional pressure $p^{(e)}/(\mu_e U/a)$ on fluid spheroid is exhibited by Fig. 7. This figure depicts that $p^{(e)}/(\mu_e U/a)$ on fluid spheroid increases with increasing cross viscosity S and deformity ε. The value of the hydrodynamic pressure on the fluid spheroid by Reiner-Rivlin fluid is higher than Newtonian fluid for higher deformation.

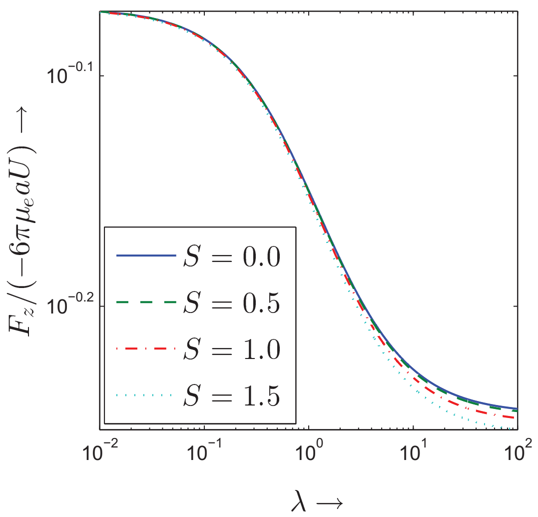

Fig. 5. The drag force on viscous droplet for different parameter S and $\varepsilon = 0.75$

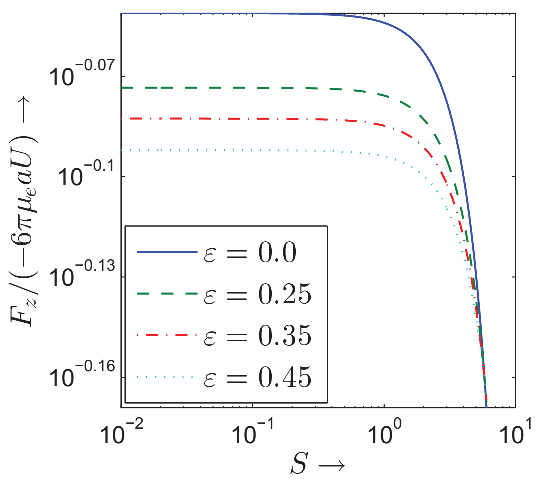

Fig. 6. The drag force on Rviscous droplet for different parameter ε and $\lambda = 0.5$

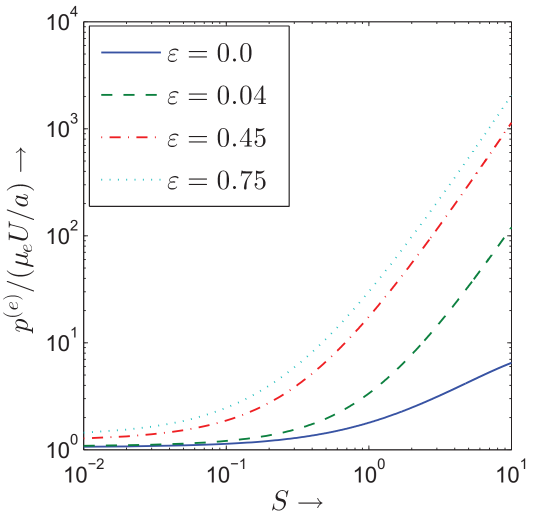

Fig. 7. The hydrodynamic pressure on viscous droplet for different parameter ε and $\lambda = 0.75$

6. Conclusion

An analytical investigation for steady flow of Newtonian drop approximated to a deformed spherical particle embedded in an infinite expanse of Reiner-Rivlin fluid is presented. The constitutive equations of the flow fields yield to momentum equations which on transformation and simplification get reduced to highly non-linear PDEs and, hence, solved using Stokes approximation. Analytical outcomes regarding drag and other fluid parameters of interest are depicted through sketches for numerous values of deformity ε, comparative viscosity λ and cross-viscous parameter S.

It is observed how various parameters influence the flow characteristics. It is noted that drag force is diminishing with respect to cross-viscous parameter (S) and relative viscosity (λ). Some constraining instances of deformed spherical body have been independently explored, and recently secured outcomes of Stokes' drag, and revision to drag force on fluid sphere by Ramkissoon's outcome [25] have been reasoned to approve our model. In the exploration, we have also found that the pressure of Reiner-Rivlin liquid on the spheroid increments both S and ε increase, whereas drag on spheroid diminishes S and ε.

As discussed, the present exploration yields explicit, analytical solution for the flow of a non-Newtonian Reiner–Rivlin fluid streaming over an approximate deformed Newtonian fluid sphere using power expansion method. To the best of author's knowledge neither Homotopy Analysis Method (HAM) nor any numerical solution exists in literature for the considered flow problem for Reiner-Rivlin fluid or for any other associated geometry. Thus, it can be concluded that there are numerous possibilities for research communities taking different geometries and methods to analyze many other non-linear problems of scientific and engineering significance considering Reiner-Rivlin fluid.

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