Applied and Computational Mechanics 16 (2022) 87–100

From A.Tondl's Dutch contacts to Neimark-Sacker-bifurcation

T. Bakri[a,b], F. Verhulst[a,*]

[a] Mathematisch Instituut, University of Utrecht, PO Box 80.010, 3508 TA Utrecht, The Netherlands
[b] TNO Sustainable Urban Mobility & Safety, PO Box 96800, 2509 JE The Hague, The Netherlands

Received 26 July 2022; accepted 13 September 2022

Abstract

We present a description of the many contacts of A. Tondl with Dutch scientists involving nonlinear dynamics models for mechanics. One of the topics is Neimark-Sacker bifurcation that leads to the presence of families of quasi-periodic solutions that are geometrically organised and visualised in tori. A new model in the spirit of A. Tondl, containing interaction of self-excited and parametrically excited oscillators is analysed to find this bifurcation and quasi-periodic solutions. The analysis using averaging in combination with numerical bifurcation tools MATCONT and AUTO produces a picture of rich dynamical phenomena with several surprises among which a special quasi-periodic solution produced by the averaged equation.

Keywords: Tondl, Neimark-Sacker bifurcation, parametric excitation, self-excitation, quasiperiodic solution

1. Introduction

Aleš Tondl (1925–2015) was born in Znojmo (Moravia, Czech Republic) where his father was headmaster of the local primary school. His scientific education took place in Brno and Prague, a considerable part of his career was spent as senior researcher at the National Research Institute for Machine Design in Běchovice (near Prague).

As a scientist Aleš Tondl was very productive but, although he experienced strong political pressure after 1968, his communication skills, hospitality and his contacts as "gentleman scientist in rough times" made him also very influential in theoretical mechanics.

A long paper or even a biography on Tondl's achievements and influence would be a suitable enterprise but we restrict ourselves here to mentioning a few of his monographs, see Section 2, and as a small but considerable part Tondl's influence on nonlinear dynamics applications in the Netherlands, Section 3. In Section 4, we describe the phenomenon of Neimark-Sacker bifurcation that gives rise to almost- or quasi-periodic solutions that are organised on tori surrounding stable or unstable periodic solutions. In Section 5, we present examples of this bifurcation in the case of interaction of self-excited and parametric excitation, a topic typical for Aleš Tondl's research.

2. Monographs

Around 1960 Tondl suggested to publish a series of books by the Běchovice Research Institute. This Institute was founded to provide a theoretical background and support for industry in Czechoslovakia. A book series would help to give the institute publicity among the community

*Corresponding author. Tel.: +31 030 253 15 26, e-mail: f.verhulst@uu.nl.
https://doi.org/10.24132/acm.2022.770

Fig. 1. Aleš Tondl in discussion with Yuri A. Kuznetsov after the thesis defence of T. Bakri, 2007

of engineers and maybe it would bring in some income for the institute. The first monograph (nr. 1, 1961) was written by him on self-excitation phenomena of rotors. Of the 10 monographs by Tondl in this series on models describing nonlinear phenomena in engineering, we mention the topics: domains of attraction of nonlinear systems, resonance problems in nonlinear systems, rotor dynamics, interaction of self-excited and parametric excitation (see Section 5), interaction of self-excitation and forced vibrations, vibrations of rigid rotors, dynamics of pump-turbines, compression or centrifugal pump systems.

His books in the Běchovice series emphasise modeling of mechanical systems with the analysis of periodic solutions and their stability; many figures illustrate the theory.

Other monographs:

- Aleš Tondl, *Some Problems in Rotor Dynamics*, Chapman & Hall, London, 1966.
- Günter Schmidt and Aleš Tondl, *Nonlinear Vibrations*, Akademie-Verlag, Berlin, 1986.
- Aleš Tondl, *Quenching of Self-Excited Vibrations*, Academia Prague, 1991.
- Aleš Tondl, Thijs Ruigrok, Ferdinand Verhulst and Radoslav Nabergoj, *Autoparametric Resonance in Mechanical Systems*, Cambridge University Press, 2000.

3. Contacts in the Netherlands

Aleš Tondl had many international contacts resulting in several scientific cooperations. A far from exhaustive list includes long-standing relations in Moscow, Vienna and Trieste. In this section, we will restrict us to cooperation on dynamical systems in the Netherlands.

The second author (F. Verhulst) met dr. Tondl at the 8th International Conference on Nonlinear Oscillations in Prague, 1978. Following this first meeting and the ensuing discussions the visits to the University of Utrecht started, going on for more than 25 years. Apart from occasional lectures in Utrecht and the Technical University of Delft, the main purpose of the cooperation was the discussion of applications of nonlinear dynamics in engineering. For the Utrecht Science Faculty, the focus in applied mathematics was on the development of analytic and numerical methods. Relatively simple examples were often used to illustrate new and old methods but Aleš Tondl introduced us to complex engineering models. These models involved new mathematical problems and complicated phenomena, producing exciting and useful new research topics. This collaboration is an excellent example of how theoretical mathematical results and insight together with exchange of ideas on actual engineering problems can be inspiring and fruitful.

Section 5 will illustrate this by new results. In the present section, we will summarise some of the work that is contained in Ph.D. theses influenced by these exchanges. Nearly all thesis chapters can be found as published papers in international journals:

- Thijs Ruigrok, *Studies in parametric and autoparametric resonance*, Ph.D. thesis, University of Utrecht, 1995.

One chapter is on rotating rigid bodies with oscillating suspension point. It has a follow-up in work with Igor Hoveijn as the problem involves versal deformation of matrices and the Whitney umbrella bifurcation, see [9]. This is connected with forced coupled oscillators in sum resonance. It is shown that certain autoparametric systems with self-excitation have the possibility of Shilnikov bifurcations leading to an infinite number of unstable periodic solutions and chaos. Part of the thesis was an inspiration for writing [18], see also [14] and [15].

- Siti Fatimah, *Bifurcations in dynamical systems with parametric excitation*, Ph.D. thesis, University of Utrecht, 2002.

A two degrees-of-freedom system is considered in 1:1 resonance with one mode in 1:2 parametric resonance. Applying averaging and numerical bifurcation analysis by CONTENT one finds periodic and quasi-periodic solutions. Of special interest is that the averaged system admits cascades of period doublings leading to chaotic solutions. In a second paper, the analysis is extended by global perturbations methods that show a Shilnikov homoclinic orbit giving rise to another type of chaotic dynamics. A third paper studies the suppression of flow-induced vibrations, where Raleigh self-excitation is suppressed by coupling to a parametrically excited oscillator. See [7] and [8].

- Abadi, *Nonlinear dynamics of self-excitation in autoparametric systems*, Ph.D. thesis, University of Utrecht, 2003.

The first paper focuses on the interaction of a Raleigh self-excited oscillator coupled to an autonomous damped oscillator. The self-excited oscillator can be destabilised leading to periodic solutions and a heteroclinic robust orbit; this cycle connects two saddle-sink equilibria. In a second paper, the interaction of a dry-friction oscillator with an autonomous damped oscillator is considered. This case involves sliding periodic solutions and a non-smooth invariant manifold. Another paper studies the problem of coupling a relaxation oscillator to an autonomous damped oscillator. Quenching of the relaxation oscillations is more difficult because of its large Lyapunov exponent. It turns out that using generalised Liénard coordinates a certain measure of quenching is possible. See [1] and [19].

- Taoufik Bakri, *Averaged behaviour of nonconservative coupled oscillators*, Ph.D. thesis, University of Utrecht, 2007.

This thesis continues a modern trend in quantitative and qualitative analysis. Analysis by averaging or other analytic methods clarifies the part played by parameters, it pinpoints elementary bifurcations. This can then be followed by numerical bifurcation programs like AUTO [6] and MATCONT [12] providing more details and illustrations. The emphasis is on the study of two and three coupled dissipative oscillators. In the first system, we have a parametrically excited oscillator nonlinearly coupled to a damped autonomous oscillator. By averaging we find periodic solutions, numerical bifurcation analysis produces a torus that by continuing on looses its smoothness in a homoclinic structure. Another chapter describes multi-frequency oscillations and torus breakdown in six-dimensional space (a few more details are given in Section 5.1), see also [2–5].

4. Tori created by Neimark-Sacker bifurcation

The creation of a torus in phase-space is of special interest in engineering, as such a torus is usually covered by a family of almost- or quasi-periodic solutions. For this phenomenon to happen, one needs at least three dimensions, it can for instance arise from a two-dimensional oscillator with forcing or two coupled oscillators.

An important scenario to create a torus arises from Neimark-Sacker bifurcation. This is a higher dimensional analogue of the Hopf bifurcation, where a periodic solution may emerge if an equilibrium contains purely imaginary eigenvalues. A well-known example is found in the Van der Pol equation. As a periodic solution may be produced from a bifurcation of an equilibrium with two double imaginary eigenvalues, in a similar way a periodic solution with two double imaginary eigenvalues may produce a torus. This is the case of a Neimark-Sacker bifurcation, also called Hopf-Hopf bifurcation. For an instructive and detailed introduction, see [10].

Apart from numerical tools such as AUTO [6] and MATCONT [12], an analytic tool such as averaging (see [16]) can be useful. Suppose that we have obtained a variational equation of the form $\dot{y} = \varepsilon f(t, y, a)$ by variation of constants and from this we have obtained an averaged equation $\dot{x} = \varepsilon f(x, a)$ with dimension three or higher; a is a parameter or a set of parameters. It is well-known that if the averaged equation contains a critical point with nonsingular Jacobian determinant, the original variational equation contains a periodic solution. This result is called the second Bogoliubov theorem. The first order approximation of this periodic solution is characterised by the timelike variables t and εt, the solution is ε-close to the critical point.

Suppose now that by varying the parameter a, a pair of eigenvalues of the critical point of the averaged system becomes purely imaginary. For this value of a, the averaged equation may experience again a Hopf bifurcation producing a periodic solution of the averaged equation. The typical time-like variable of this periodic solution is εt and so the period will be $O(1/\varepsilon)$. As it branches off, an existing periodic solution in the original equation, it will produce a torus. It is associated with a Hopf bifurcation of the corresponding Poincaré map and the bifurcation has a different name: Neimark-Sacker bifurcation. The result will in the three- or four-dimensional case be a two-dimensional torus, which contains two-frequency oscillations, one on a timescale of order one and the other on timescale $O(1/\varepsilon)$. A two degrees-of-freedom example is discussed in [4], where also comparison is made with the harmonic balance method. The changes of tori by varying a parameter leading to torus breakdown is studied in [5].

5. Interaction of self-excited and parametric excitation

One of Tondl's many interests was concerned with the interaction of self-excited and parametric vibrations, see [17]. We will sketch the results of a complex model published in [2]. After this we will present new results for a model with a two degrees-of-freedom (2 DOF) interaction.

5.1. A model for quenching flow-induced vibrations

In a private communication of Tondl in 2002, a model is given for quenching undesirable flow-induced vibrations that might be influenced or reduced by coupling to parametric vibration absorbers. The system for the deflections y_1, y_2, y_3 is

$$\begin{aligned} m_1\ddot{y}_1 + b\dot{y}_1 + k_0(1+\varepsilon\cos\omega t)y_1 - k_1(y_2 - y_1) &= 0,\\ m_2\ddot{y}_2 + \beta U^2(1-\gamma\dot{y}_2^2)\dot{y}_2 + 2k_1y_2 - k_1(y_1+y_3) &= 0,\\ m_3\ddot{y}_3 + b\dot{y}_3 + k_0(1+\varepsilon\cos\omega t)y_3 - k_1(y_2 - y_3) &= 0. \end{aligned} \tag{1}$$

All parameters are assumed to be positive, the flow-induced vibrations of mass m_2 with flow strength U are caused by a so-called Raleigh self-excitation term. The idea is to quench the flow-induced vibrations of mass m_2 by nonlinear coupling to two parametrically excited oscillators with masses m_1, m_3. The analysis is complicated and leads to periodic solutions that by Neimark-Sacker bifurcation evolve to tori. Slowly changing the parameters in this three-mass system, one finds break-up of the tori and evolution to strange attractors. This is basically an instructive example of the Ruelle-Takens scenario [13] that sketches the origin of turbulence by such a sequence of bifurcations from unstable periodic solutions first to tori and then to strange attractors. For extensive details of this flow-induced vibration model, see [2]. Omitted in [2] is the dynamics of the case $m_1 = m_3$ as this leads to a 2 DOF system while the emphasis in [2] is on the much more complicated 3 DOF case.

5.2. A typical simplified model

We will study in more detail a simplified model with linear interaction between the parametrically- and self-excited modes. The system is

$$\begin{aligned} \ddot{x} + \varepsilon\kappa\dot{x} + (1+\varepsilon\cos 2t)x - \varepsilon c y &= 0,\\ \ddot{y} - \varepsilon\mu(1-y^2)\dot{y} + y - \varepsilon c x &= 0. \end{aligned} \tag{2}$$

The parameters κ, c, μ will be specified later but κ, $\mu > 0$. In the two modes, we recognise respectively parametric excitation and self-excited vibrations, the interaction has a simple form that excludes the existence of x and y normal modes if $c \neq 0$.

Qualitative changes in the system are signalled by bifurcations. We will be especially interested in Neimark-Sacker bifurcations that produce quasi-periodic solutions. Other bifurcation will come up with notations like 'fold' or 'LPC'. They play not an important part here, for details about such bifurcations, see [10].

For general position interactions, we use new, slowly varying variables r, ψ using the transformation

$$\begin{aligned} x &= r_1(t)\cos(t+\psi_1(t)), \quad \dot{x} = -r_1(t)\sin(t+\psi_1(t)),\\ y &= r_2(t)\cos(t+\psi_2(t)), \quad \dot{y} = -r_2(t)\sin(t+\psi_2(t)). \end{aligned}$$

If $\varepsilon = 0$, the amplitudes r and phases ψ are constant. We use variation of constants to find variational equations for amplitudes r_1, r_2 and phases ψ_1, ψ_2 for $\varepsilon > 0$. We leave out the variational equations. After averaging these equations over time we find

$$\begin{aligned} \dot{r}_1 &= \frac{\varepsilon}{2}\left(-\kappa r_1 + \frac{1}{2}r_1\sin 2\psi_1 - c r_2\sin\chi\right),\\ \dot{\psi}_1 &= \frac{\varepsilon}{2r_1}\left(\frac{1}{2}r_1\cos 2\psi_1 - c r_2\cos\chi\right),\\ \dot{r}_2 &= \frac{\varepsilon}{2}\left[\mu r_2\left(1-\frac{1}{4}r_2^2\right) + c r_1\sin\chi\right],\\ \dot{\psi}_2 &= -\frac{\varepsilon}{2}c\frac{r_1}{r_2}\cos\chi, \end{aligned} \tag{3}$$

with $\chi = \psi_1 - \psi_2$.

5.2.1. Equilibria and periodic solutions

The averaged system has general position equilibria (i.e., critical points of the vector field) if $\cos\chi = 0$ or $\sin\chi = \pm 1$. From the equation for ψ_1, it follows that in this case $\cos 2\psi_1 = 0$ and $\sin 2\psi_1 = \pm 1$. The other two conditions for critical points are

$$-\kappa r_1 + \frac{1}{2} r_1 \sin 2\psi_1 - c r_2 \sin\chi = 0, \qquad \mu r_2 \left(1 - \frac{1}{4} r_2^2\right) + c r_1 \sin\chi = 0. \tag{4}$$

We have various cases for the critical values

$$r_1 = \frac{c\sin\chi}{-\kappa + 0.5\sin 2\psi_1} r_2, \qquad r_2 = 2\sqrt{1 + \frac{c^2}{\mu(-\kappa + 0.5\sin 2\psi_1)}}. \tag{5}$$

Non-trivial equilibria can be found for $\sin\chi = 1$, $c < 0$, $\sin 2\psi_1 = -1$, $\kappa > 0$ and $\mu > 0$ large enough. Equation (5) becomes in this case

$$r_1 = \frac{c}{-\kappa - 0.5} r_2, \qquad r_2 = 2\sqrt{1 + \frac{c^2}{\mu(-\kappa - 0.5)}}. \tag{6}$$

The Jacobian matrix becomes at the non-trivial equilibrium

$$\begin{bmatrix} -\kappa - \frac{1}{2} & 0 & -c & 0 \\ 0 & \frac{1}{2} - \kappa & 0 & \kappa + \frac{1}{2} \\ c & 0 & \frac{6c^2}{2\kappa+1} - 2\mu & 0 \\ 0 & -\frac{2c^2}{2\kappa+1} & 0 & \frac{2c^2}{2\kappa+1} \end{bmatrix} \tag{7}$$

and will give us information about possible periodic solutions corresponding with the critical points. In this case, the eigenvalues of the matrix can be computed analytically by MATHEMATICA. We find

$$\lambda_1 = \frac{-\sqrt{-8c^2[4\kappa(\kappa+2)+3] + 16c^4 + (1-4\kappa^2)^2} + 4c^2 - 4\kappa^2 + 1}{8\kappa + 4}, \tag{8}$$

$$\lambda_2 = \frac{\sqrt{-8c^2[4\kappa(\kappa+2)+3] + 16c^4 + (1-4\kappa^2)^2} + 4c^2 - 4\kappa^2 + 1}{8\kappa + 4}, \tag{9}$$

$$\lambda_3 = \frac{-\sqrt{8c^2(2\kappa-1)(2\kappa-12\mu+5) + 144c^4 + (1-2\kappa)^2(2\kappa-4\mu+1)^2} + R}{8\kappa - 4}, \tag{10}$$

$$\lambda_4 = \frac{\sqrt{8c^2(2\kappa-1)(2\kappa-12\mu+5) + 144c^4 + (1-2\kappa)^2(2\kappa-4\mu+1)^2} + R}{8\kappa - 4}, \tag{11}$$

$$R = 12c^2 - 4\kappa - 4\kappa^2 - 8\kappa\mu - 4\mu - 1. \tag{12}$$

It is easy to see from (8) and (9) that the real part (i.e., the part outside the square root symbol) of λ_1 and λ_2 becomes zero when $c = -\frac{1}{2}\sqrt{-1 + 4\kappa^2}$ with $\kappa \geq \frac{1}{2}$. Along this curve in parameter space, the term inside the square root reduces to $-4\kappa^3 - 2\kappa^2 + k + \frac{1}{2}$. This term is negative for $\kappa > \frac{1}{2}$. This means that the Hopf curve in the averaged system and consequently the Neimark-Sacker bifurcation curve in the original system leading to torus dynamics is given by the closed form

$$c(\kappa) = -\frac{1}{2}\sqrt{-1 + 4\kappa^2}. \tag{13}$$

Along this curve the eigenvalues λ_1 and λ_2 are purely imaginary. Adding a small perturbation $0 < \delta \ll 1$ to c yields a transversal crossing of the imaginary axis and hence a Hopf bifurcation. Note that this curve is independent of the self-excitation parameter μ. A second Hopf bifurcation occurs as well when the term R becomes zero. Requiring the conditions for the parameters above (i.e., $c < 0$, κ, $\mu > 0$, gives the analytical closed form for the second Hopf curve

$$c(\kappa, \mu) = -\frac{1}{2}\sqrt{\frac{1 + 4\kappa + 4\mu + 4\kappa^2 + 8\kappa\mu}{3}}. \tag{14}$$

Numerical analysis below will reveal that the first Hopf bifurcation is supercritical yielding a stable torus in the original system. The second Hopf bifurcation is subcritical yielding an unstable torus in the original system. An example of one of the periodic solutions emerging from the Hopf bifurcations is given in Fig. 2.

The second non-trivial equilibrium of system (3) can be found for $\sin\chi = 1$, $c < 0$, $\sin 2\psi_1 = 1$, $\kappa > 0.5$ and $\mu > 0$ large enough. Equation (5) for the critical amplitudes becomes in this case

$$r_1 = \frac{c}{-\kappa + 0.5} r_2, \qquad r_2 = 2\sqrt{1 + \frac{c^2}{\mu(-\kappa + 0.5)}}. \tag{15}$$

Stability analysis of the second non-trivial equilibrium that corresponds with a periodic solution in the original system is elementary and straightforward. This makes the averaging method for small values of ε a powerful and easy to use tool to study the stability and bifurcations of periodic orbits without the use of sophisticated continuation software packages. The continuation toolboxes will come into play later on when we want to follow the emerging tori for larger values in the parameter space; with these toolboxes we also compute the first order normal form coefficients to establish stability for larger values of ε. Choosing for instance $\varepsilon = 1$, the periodic solution still exists, the analytical expressions of the Jacobian matrix and its eigenvalues

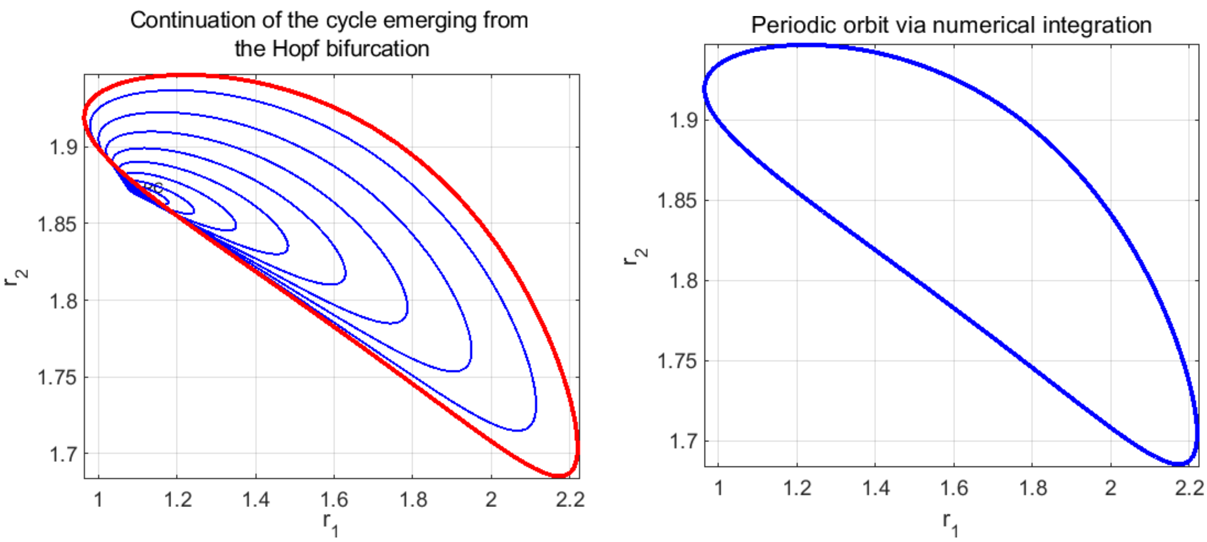

Fig. 2. (*Left*) A stable periodic solution of system (3) obtained using MATCONT by continuation of the orbit emerging from the supercritical Hopf bifurcation using the parameter c as a control parameter. The red orbit corresponds with the parameter values $c = -1$, $k = 1$ and $\mu = 4$. (*Right*) We get the same orbit by numerical integration. We took the parameter values of the red orbit with initial condition starting at $(r_1, \phi_1, r_2, \phi_2) = (1.117\,35, 2.705\,16, 1.944\,95, -0.018\,881\,9)$

are given below

$$\begin{bmatrix} \frac{1}{4}(1-2\kappa) & 0 & -\frac{c}{2} & 0 \\ 0 & \frac{1}{4}(-2\kappa-1) & 0 & \frac{1}{4}(2\kappa-1) \\ \frac{c}{2} & 0 & \frac{3c^2}{2\kappa+1}-\mu & 0 \\ 0 & \frac{c^2}{1-2\kappa} & 0 & \frac{c^2}{2\kappa-1} \end{bmatrix}. \tag{16}$$

In this case, the eigenvalues at the second equilibrium can also be computed analytically using MATHEMATICA. We find

$$\lambda_1 = \frac{-\sqrt{-8c^2[4(\kappa-2)\kappa+3]+16c^4+(1-4\kappa^2)^2}+4c^2-4\kappa^2+1}{16\kappa-8}, \tag{17}$$

$$\lambda_2 = \frac{\sqrt{-8c^2[4(\kappa-2)\kappa+3]+16c^4+(1-4\kappa^2)^2}+4c^2-4\kappa^2+1}{16\kappa-8}, \tag{18}$$

$$\lambda_3 = \frac{-\sqrt{8c^2(2\kappa-1)(2\kappa-12\mu-1)+144c^4+(1-2\kappa)^2(-2\kappa+4\mu+1)^2}+R}{16\kappa-8}, \tag{19}$$

$$\lambda_4 = \frac{\sqrt{8c^2(2\kappa-1)(2\kappa-12\mu-1)+144c^4+(1-2\kappa)^2(-2\kappa+4\mu+1)^2}+R}{16\kappa-8}, \tag{20}$$

$$R = 12c^2-(2\kappa-1)(2\kappa+4\mu-1). \tag{21}$$

Requiring the part outside the square root symbol of the eigenvalues λ_1, λ_2 to be zero as a necessary condition for a Hopf bifurcation of the second equilibrium does not yield a Hopf bifurcation. One can easily show that in this case the eigenvalues λ_1, λ_2 are real with opposite signs. A Hopf bifurcation does occur when requiring the part outside the square root symbol of the eigenvalues λ_3, λ_4 to become zero. In this case, the Hopf curve can again be computed analytically. We find

$$c(\kappa,\mu) = -\frac{1}{2}\sqrt{\frac{(2\kappa-1)(2\kappa+4\mu-1)}{3}}. \tag{22}$$

Numerical analysis using the bifurcation analysis toolbox MATCONT reveals a subcritical Hopf bifurcation yielding an unstable torus in the original system. This equilibrium will, therefore, be left out of the bifurcation analysis below. In this paper, we focus on invariant tori that are asymptotically stable and, therefore, easily detectable by numerically integration of the vector field of the original system.

5.2.2. *Numerical examples of the simplified model*

Taking $\kappa = 1$, $\mu = 4$ and $c = -\frac{1}{2}\sqrt{-1+4\kappa^2} = -\frac{\sqrt{3}}{2} \doteq -0.866\,025\,4$ along the Hopf curve as stated above, we have a stable equilibrium with $r_1 = 1.080\,12$ and $r_2 = 1.870\,83$ and the following eigenvalues:

$$\Lambda = \left\{-\frac{i}{2\sqrt{2}}, \frac{i}{2\sqrt{2}}, \frac{1}{4}\left(-8-\sqrt{22}\right), \frac{1}{4}\left(-8+\sqrt{22}\right)\right\}. \tag{23}$$

We check the transversality of the bifurcation. Adding a small perturbation $\delta = 0.01$ to the parameter c and keeping the other parameters constant gives the following eigenvalues:

$$\Lambda = \{-0.002\,870\,08-0.349\,459i, -0.002\,870\,08+0.349\,459i, -3.192\,21, -0.825\,012\}. \tag{24}$$

Subtracting δ from c yields the following eigenvalues:

$$\Lambda = \{0.002\,903\,42 - 0.357\,624i, 0.002\,903\,42 + 0.357\,624i, -3.152\,73, -0.829\,849\}\,. \quad (25)$$

The eigenvalues λ_1 and λ_2 cross transversely the imaginary axis as the critical interaction parameter c crosses the value $-\frac{\sqrt{3}}{2}$ and, therefore, we establish almost by hand the occurrence of a Hopf bifurcation of the equilibrium in question. Numerical analysis by MATCONT with the parameter c as a control parameter yields a supercritical Hopf bifurcation (normal form coefficient $l_1 = -0.025\,670\,407\,5$) at the parameter values $c = -0.866\,034$, which is in good agreement with the analysis above. This shows that MATCONT pinpoints the bifurcation value with very good accuracy.

From the normal form coefficient obtained by MATCONT, we know that the emerging torus in the original system is asymptotically stable. In Fig. 2, we show that the emerging cycle is indeed stable as was found also by the numerical integration.

5.2.3. Bifurcation analysis of the averaged system

Various software toolboxes are available to perform a bifurcation analysis. Each of them has advantages and drawbacks. For a concise comparison of the available bifurcation software, we refer to [11]. In the following analysis, we made mainly use of the software packages AUTO and MATCONT. AUTO is very fast and needs very little resources to run smoothly. It even runs on single-board computers like the Raspberry pi. It gives at low cost a rough but accurate view of the dynamics and bifurcations involved in the system under study. One of its drawbacks is its incapacity of computing normal forms coefficients. MATCONT, on the other hand, does detect more higher co-dimension bifurcations and computes normal form coefficients, but it is slow and needs the proprietary software MATLAB to be installed to run on a machine. Here we combine the power of both.

As can be seen from Fig. 3, the non-trivial equilibrium of the averaged system (3) is continued starting at the parameter values $\kappa = 1.3$, $\mu = 4$, and $c = -1$, and using the parameter c as the only control parameter. A Hopf supercritical bifurcation point labeled (H) is detected at $c = -1.2$ as predicted by (13). A codimension 2 Hopf curve (Hopf+) is obtained from point (H) by continuation using the parameter κ as a second control parameter. As expected, MATCONT as well as AUTO compute the Hopf+ curve with very high precision. It fits exactly the formula given by (13). The unstable equilibrium undergoes a second (sub-critical) Hopf bifurcation at $c = -2.424\,871\,2$. Using κ as a second control parameter, the Hopf– curve is generated. The numerical results here are again in complete agreement with (14). The stable cycle emerging from the Hopf bifurcation at $c = -1.2$ has a period $T = 14.05$. It is continued using the parameter c as the control parameter. The cycle undergoes two branching point cycle bifurcations at resp. $c = -1.531\,603\,4$ and $c = -1.690\,769\,1$. At these critical values, a second cycle branches off of the stable cycle yielding two more periodic orbits in the averaged system and, thus, two $\mathbb{T}^2$ tori in the original system. The cycle undergoes also a (subcritical) Neimark-Sacker bifurcation at $c = -1.697\,759\,3$ yielding an unstable $\mathbb{T}^2$ torus in the averaged system and hence an unstable $\mathbb{T}^3$ in the original system (2). The Neimark-Sacker (NS) curve (red in Fig. 3) is obtained in MATCONT by continuation of the NS bifurcation with κ as a second control parameter. Interesting bifurcations take place on the NS curve like strong resonances R3 and R4, a Chenciner bifurcation was also detected. These bifurcations are outside the scope of this paper and will, therefore, be left out of the analysis. The branching cycle at $c = -1.531\,603\,4$ is also continued with respect to the parameter c. It undergoes a series of bifurcations. See Fig. 4 (green curve).

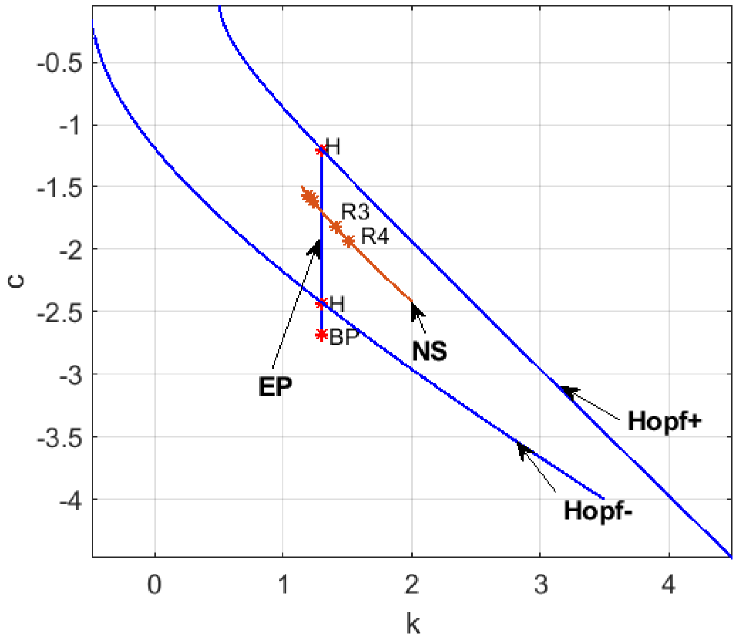

Fig. 3. (Partial) Bifurcation diagram of system (3) in the kc-parameter plane showing both Hopf curves and the Neimark-Sacker (NS) bifurcation curve with the strong resonance bifurcation points R3 and R4

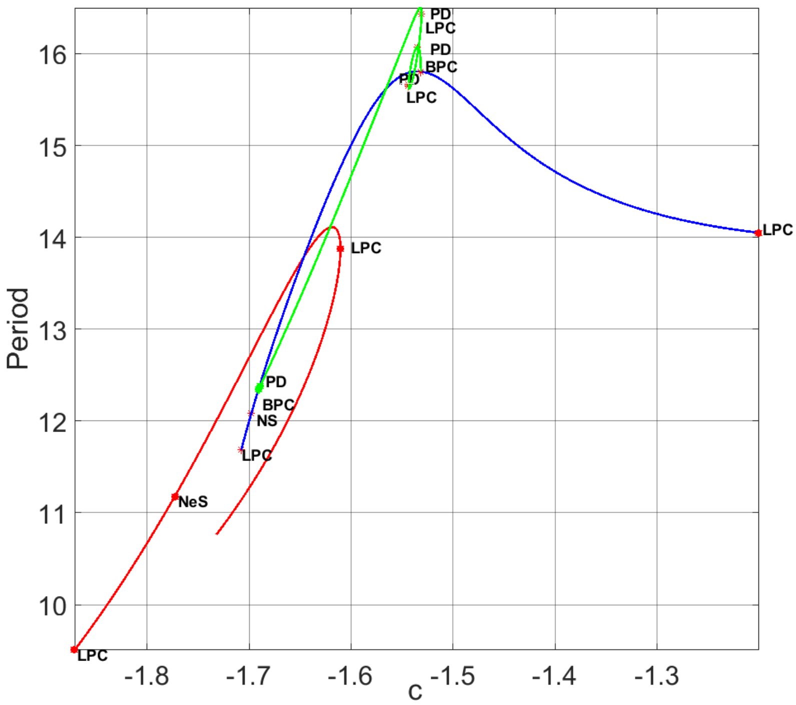

Fig. 4. Bifurcation diagram of the periodic orbit of system (26) together with the orbit emerging from the supercritical Hopf bifurcation using the parameter c as a control parameter and plotted against the period of the cycle. PD stands for Period doubling bifurcation, BPC stands for Branching Point Cycle bifurcation, LPC indicates a Limit Point Cycle or fold bifurcation and NeS is used to label a Neutral Saddle bifurcation

5.2.4. *Bifurcations of a special quasi-periodic solution*

According to the second Bogoliubov theorem, equilibria of the averaged system (3) correspond with periodic solutions of the original system (2). The proof of the theorem is based on the implicit function theorem with the implication that for instance amplitudes and period are close to these quantities of system (2) with $\varepsilon = 0$. The averaged system, and so the original system, may also contain solutions that cannot be continued from the unperturbed solutions. Surprisingly, the averaged system turns out to have a stable periodic orbit that is not related to any of the equilibria found above. As we shall see, such a solution corresponds with a quasi-periodic solution of the original system.

To be certain that we are not excluding solutions close to normal modes, we will use other coordinates for the averaging analysis. We switch to the coordinates (x_1, x_2, y_1, y_2) instead of the amplitude-phase coordinates by transforming

$$\begin{aligned} x(t) &= x_1(t)\cos(t) + x_2(t)\sin(t), \quad \dot{x}(t) = -x_1(t)\sin(t) + x_2(t)\cos(t),\\ y(t) &= y_1(t)\cos(t) + y_2(t)\sin(t), \quad \dot{y}(t) = -y_1(t)\sin(t) + y_2(t)\cos(t). \end{aligned}$$

Averaging the variational equations over time yields the following system that is equivalent to system (3) outside the normal modes

$$\begin{aligned} \dot{x_1} &= -\frac{1}{4}\varepsilon(2cy_2 + 2kx_1 + x_2),\\ \dot{x_2} &= -\frac{1}{4}\varepsilon(-2cy_1 + 2kx_2 + x_1),\\ \dot{y_1} &= -\frac{1}{8}\varepsilon\left[4cx_2 + \mu y_1\left(y_1^2 + y_2^2 - 4\right)\right],\\ \dot{y_2} &= \frac{1}{8}\varepsilon\left[4cx_1 - \mu y_2\left(y_1^2 + y_2^2 - 4\right)\right]. \end{aligned} \tag{26}$$

System (26) turns out to have an asymptotically stable periodic solution that is not related to the periodic solutions found in the Hopf bifurcations discussed above, see Fig. 4. Starting at initial condition

$$(x_1, x_2, y_1, y_2) = (0.060\,412, 1.503\,658, -0.233\,880, 1.878\,305)$$

with parameter values $c = -1.732\,128$, $\mu = 4$ and $\kappa = 1.3$, a stable periodic orbit of the averaged system is found with period $T = 10.76$ (not close to 2π). In the original system (2), this produces a solution characterised by two periods, it is quasi-periodic. Numerical continuation of this orbit with respect to parameter c yields the red curve in Fig. 4. The periodic orbit undergoes a fold bifurcation also known as a *Limit Point Cycle* (LPC) bifurcation at parameter value $c = -1.610\,449\,5$. At this point, the periodic orbit collides with an unstable cycle and disappears. Following the unstable cycle, it undergoes a Neutral Saddle bifurcation (NeS) at $c = -1.772\,976\,3$ and another LPC bifurcation at $c = -1.871\,089\,1$.

5.2.5. *The origin of the quasi-periodic solution*

A simple way to understand where this special solution in the averaged system comes from is to consider the parameter c to be large (or taking εc as $O(1)$ quantity). Putting for the parameter $c = \tilde{c}/\varepsilon$, omitting the tilde for notational simplicity and keeping the other parameter $O(1)$ with respect to ε, system (2) can be transformed into a quasi-normal form

$$\begin{aligned} \ddot{x}_1 + \tfrac{c^2}{4}x_1 &= \varepsilon\left(\frac{-2\dot{x}_1\left(c^2(\kappa-\mu) + \dot{x}_2^2\mu - c^2\dot{x}_2 - 2\dot{x}_1^3\mu\right)}{4c^2}\right) + O(\varepsilon^2), \\ \ddot{x}_2 + \tfrac{c^2}{4}x_2 &= \frac{1}{16}\varepsilon\left(-\frac{8\dot{x}_2\left(c^2(\kappa-\mu) + \mu\left(\dot{x}_1^2 + \dot{x}_2^2\right)\right)}{c^2} - 4\dot{x}_1\right) + O(\varepsilon^2). \end{aligned} \tag{27}$$

Here, the variables x_1 and x_2 are exactly the same as used for the averaged system (26).

Looking for periodic solutions of system (26) is equivalent to looking for periodic orbits of system (27) so we apply averaging and look for non-trivial equilibria. As we know now that the normal modes are not involved, we can use slowly varying phase-amplitude variables

$$x_1 = r_1(t)\cos\left(\frac{c}{2}t + \phi_1(t)\right), \qquad \dot{x}_1 = -\frac{c}{2}r_1(t)\sin\left(\frac{c}{2}t + \phi_1(t)\right),$$
$$x_2 = r_2(t)\cos\left(\frac{c}{2}t + \phi_2(t)\right), \qquad \dot{x}_2 = -\frac{c}{2}r_2(t)\sin\left(\frac{c}{2}t + \phi_2(t)\right)$$

to find after averaging the following system:

$$\begin{aligned} \dot{r}_1 &= -\frac{1}{64}\varepsilon\left\{r_1\left[\mu {r_2}^2\cos 2\chi + 16\kappa + 3\mu {r_1}^2 + 2\mu\left({r_2}^2 - 8\right)\right] + 8r_2\cos\chi\right\}, \\ \dot{\phi}_1 &= \frac{1}{32r_1}\varepsilon\left[r_2\sin\chi(\mu r_1 r_2\cos\chi + 4)\right], \\ \dot{r}_2 &= -\frac{1}{64}\varepsilon\left\{r_1\left[\mu r_1 r_2\cos(2\chi) + 8\cos\chi\right] + 2r_2\left[8\kappa + \mu\left({r_1}^2 - 8\right)\right] + 3\mu {r_2}^3\right\}, \\ \dot{\phi}_2 &= -\frac{1}{32r_2}\varepsilon\left[r_1\sin\chi(\mu r_1 r_2\cos\chi + 4)\right] \end{aligned} \tag{28}$$

with $\chi = \phi_1 - \phi_2$. System (28) has two non-trivial equilibria when $\sin\chi = 0$. We find

$$r_1 = r_2 = 2\sqrt{\frac{-2\kappa + 2\mu - 1}{3\mu}}, \quad \text{or} \quad r_1 = r_2 = 2\sqrt{\frac{-2\kappa + 2\mu + 1}{3\mu}}. \tag{29}$$

A third non-trivial equilibrium emerges when $\cos(\chi) = -4/(\mu r_1 r_2)$ and $-4/(\mu r_1 r_2) \in [-1, 0]$. We find for r_1 and r_2 the following expression:

$$r_1 = r_2 = 2\sqrt{\frac{\mu - \kappa}{\mu}}. \tag{30}$$

We conclude that system (26) has 3 periodic orbits with period $T = -4\pi/c$ and amplitudes as given above.

5.2.6. Comparison of averaging results with the numerics

Substituting the numerical values of the parameters used in the continuation (i.e., $\mu = 4$, $\kappa = 1.3$) in the equations above, we find the three periodic solutions of the averaged system as predicted by the averaging method; they have respectively the amplitudes: $r_1 = r_2 = 1.21$, $r_1 = r_2 = 1.49$ and $r_1 = r_2 = 1.64$. The periodic solution of system (26) numerically found turns out to coincide with the third periodic orbit with amplitudes $r_1 = r_2 = 1.64$. Using MATCONT the stable periodic orbit in Fig. 5 was continued for large values of the parameter c. The periodic orbit found numerically has an amplitude that tends to 1.64 as the parameter c tends to $-\infty$. This is in good agreement with the analytical results obtained above, see Figs. 6 and 7.

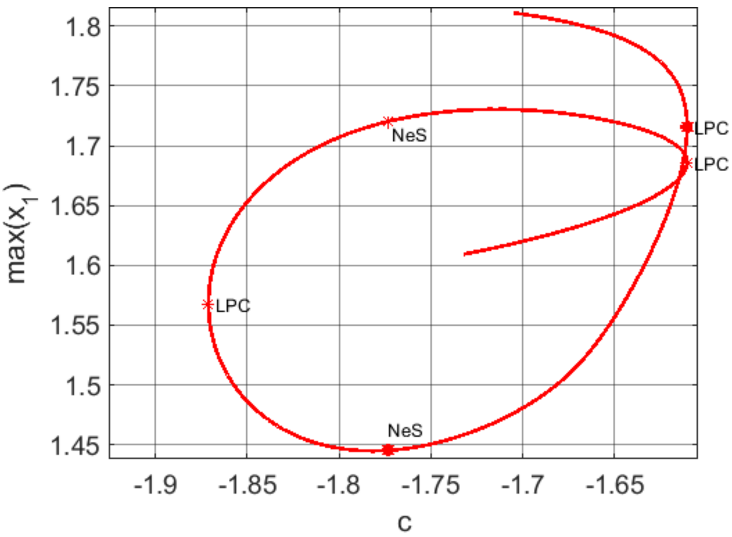

Fig. 5. Bifurcation diagram of the periodic orbit of system (26) as a function of the parameter c plotted against the $\max(x_1)$

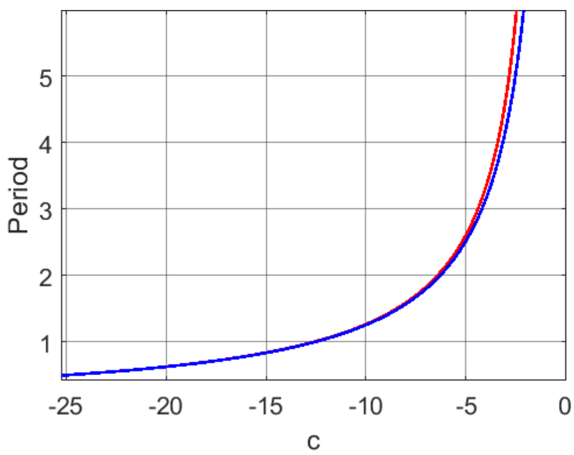

Fig. 6. Plot of the numerically computed period (red) and the analytically predicted period $T = -4\pi/c$ (blue) of system (26) for large negative values of the parameter c

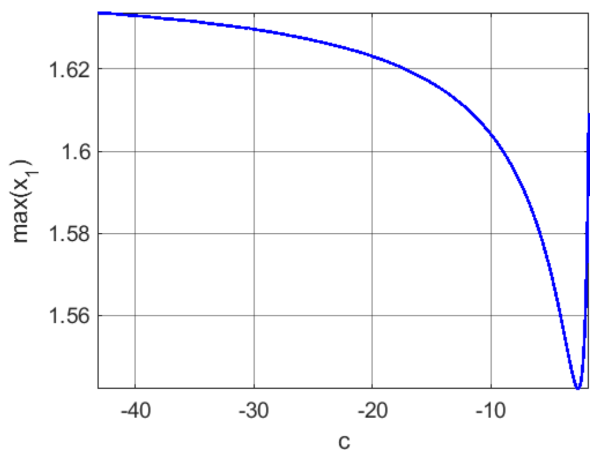

Fig. 7. Plot of the numerically computed amplitude $\max(x_1)$ of system (26) as function of c for large negative values of the parameter c

6. Conclusion for the simplified system

The simplified system (2) contains self-excitation, parametric excitation and mode interaction in its simplest form. It is remarkable that for this system a large number of bifurcational phenomena can be found, with analytic and numerical methods complementing each other. The averaging method and the numerical tools AUTO, MATCONT were used to produce basic results.

Acknowledgement

Editorial suggestions by dr. Luděk Pešek are gratefully acknowledged.

References

[1] Abadi, On self-excited auto-parametric systems, Nonlinear Dynamics 24 (2) (2001) 147–166. https://doi.org/10.1023/A:1008328100880

[2] Bakri, T., Kuznetsov, Y. A., Verhulst, F., Torus bifurcations in a mechanical system, Journal of Dynamics and Differential Equations 27 (3-4) (2015) 371–403. https://doi.org/10.1007/s10884-013-9339-9

[3] Bakri, T., Nabergoj, R., Tondl, A., Multi-frequency oscillations in self-excited systems, Nonlinear Dynamics 48 (1-2) (2007) 115–127. https://doi.org/10.1007/s11071-006-9077-1

[4] Bakri, T., Nabergoj, R., Tondl, A., Verhulst, F., Parametric excitation in non-linear dynamics, International Journal of Non-Linear Mechanics 39 (2) (2004) 311–329. https://doi.org/10.1016/S0020-7462(02)00190-7

[5] Bakri, T., Verhulst, F., Bifurcations of quasi-periodic dynamics: Torus breakdown, ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik 65 (6) (2014) 1053–1076. https://doi.org/10.1007/s00033-013-0363-8

[6] Doedel, E., Champneys, A. R., Fairgrieve, T. F., Kuznetsov, Y. A., Sandstede, B., Wang, X. J., AUTO97: Continuation and bifurcation software for ordinary differential equations (with HomCont), Concordia University, Montreal, Canada, 1998.

[7] Fatimah, S., Ruijgrok, M., Bifurcations in an autoparametric system in 1:1 internal resonance with parametric excitation, International Journal of Non-Linear Mechanics 37 (2) (2002) 297–308. https://doi.org/10.1016/S0020-7462(00)00115-3

[8] Fatimah, S., Verhulst, F., Suppressing flow-induced vibrations by parametric excitation, Nonlinear Dynamics 31 (3) (2003) 275–297. https://doi.org/10.1023/A:1022974024810

[9] Hoveijn, I., Ruijgrok, M., The stability of parametrically forced coupled oscillators in sum resonance, ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik 46 (3) (1995) 384–392. https://doi.org/10.1007/BF01003557

[10] Kuznetsov, Y. A., Elements of applied bifurcation theory, Springer, New York, 2004. https://doi.org/10.1007/978-1-4757-3978-7

[11] Kuznetsov, Y. A., Trends in bifurcation software: From CONTENT to MATCONT, Proceedings of the 4th Workshop on Computation of Biochemical Pathways and Genetic Networks, Heidelberg, Logos Verlag Berlin, 2005, pp. 49–57.

[12] MATCONT – Numerical continuation and bifurcation program, http://www.matcont.ugent.be.

[13] Ruelle, D., Takens, F., On the nature of turbulence, Communications in Mathematical Physics 20 (1971) 167–192. https://doi.org/10.1007/BF01646553

[14] Ruijgrok, M., Tondl, A., Verhulst, F., Resonance in a rigid rotor with elastic support, ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik 73 (10) (1993) 255–263. https://doi.org/10.1002/zamm.19930731006

[15] Ruijgrok, M., Verhulst, F., Parametric and autoparametric resonance, In: Progress in Nonlinear Differential Equations and Their Applications, Vol. 19, Birkhäuser, Basel, 1996, pp. 279–298. https://doi.org/10.1007/978-3-0348-7518-9_13

[16] Sanders, J. A., Verhulst, F., Murdock, J., Averaging methods in nonlinear dynamical systems, Springer-Verlag, New York, 2007. https://doi.org/10.1007/978-0-387-48918-6

[17] Tondl, A., On the interaction between self-excited and parametric vibrations, National Research Institute for Machine Design, Běchovice, Monographs and Memoranda 25, Prague, 1978.

[18] Tondl, A., Ruijgrok, T., Nabergoj, R., Verhulst, F., Autoparametric resonance in mechanical systems, Cambridge University Press, 2000.

[19] Verhulst, F., Abadi, Autoparametric resonance of relaxation oscillations, ZAMM – Zeitschrift für Angewandte Mathematik und Mechanik 85 (2) (2005) 122–131. https://doi.org/10.1002/zamm.200410159