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Higher order computational model considering the effects of transverse normal strain and 2-parameter elastic foundation for the bending of laminated panels

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Abstract

In this study, the authors analyze laminated composite panels supported on an elastic foundation considering the effects of transverse normal strain. A 2-parameter, i.e., Winkler and Pasternak foundation model is assumed to represent the interaction between the panels and the foundation. The theory presented here takes into account the effects of transverse shear and normal strains. The theory plots realistic distributions of the transverse shear stress through the plate thickness and satisfies the shear-free conditions at the extreme surfaces of the panel. The differential equations of the present model are obtained from the principle of virtual work. The laminated composite panel resting on the elastic foundation is analyzed for simply supported boundary conditions. For the verification purpose, the presented problems are also solved using the Reddy's model, Mindlin's model, and the classical model. Good agreement is observed between the numerical results obtained using the present model and the other models.

Keywords: laminated panels, 2-parameter foundation, bending analysis, shear deformation, transverse normal strain, transverse shear stresses

1. Introduction

Since the last decade, the applications of fibrous laminated composite panels have gained popularity in different sectors of industry, such as civil, mechanical, and aerostructures, because of their good strength and stiffness properties despite being light in weight. The increasing use of laminated composite panels in various structures has attracted great interest in their structural analysis. For multilayer laminated panels, the inter-laminar shearing stresses and strains between the layers strongly influence the bending behavior under transverse loading. The effects of shearing stresses become more pronounced in shear deformable laminated panels. However, these stresses are neglected in the classical model, so this model fails to capture the exact bending behavior of thick composite panels. Therefore, shear strain theory is required to explain the correct bending behavior of thick panels, including shear strain effects and related cross-sectional deformations.

The first-order shear deformation plate theory (FSDT), developed by Mindlin [9], considers the transverse shearing effect for the first time. However, FSDT violates the zero shearing stress condition at the top and bottom of the panel and requires a correction to account for the

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strain energy caused by shearing deformation. For these reasons, researchers have developed sophisticated panel theory to describe the precise bending behavior of laminated panels.

The theory by Reddy [12] is the most used plate theory by many researchers during the last 38 years. Akavci [3] applied his third-order shear deformable model, which considers the effect of shearing strain, to the bending analysis of symmetrically laminated panels on elastic bases. Akavci et al. [1] applied Mindlin's plate model for symmetrically laminated composite panels on elastic bases. Akavci [4] published an analysis of buckling and free vibration of laminated composite panels placed on elastic foundations. Akavci [2] developed two new hyperbolic models for the analysis of laminated panels. Sayyad and Ghugal [14–16] developed a sine and cosine model that considers the effects of both shear and normal strains to examine the bending of laminated panels with and without elastic supports. Zenkour et al. [24–26] showed some problems with laminated panels on elastic foundations. Zenkour [23] presented a bending analysis of orthotropic panels on Pasternak elastic foundations using the theory of mixed shear deformation. Zenkour and Ashraf [22] presented a bending analysis of functionally graded panels resting on elastic supports. Khateeb and Zenkour [7] studied the static response of advanced composite panels placed on elastic foundations, including moisture and temperature loads. They investigated the effects of the Winkler and Pasternak fundamental parameters, temperature, moisture concentration, and volume fraction distribution, on the displacement and stresses. Setoodeh and Azizi [17] presented static and vibration analysis of $0^\circ/90^\circ$ and $\theta/-\theta$ laminate composite panels on elastic foundations using the four-variable laminate panel theory. Nedri et al. [11] presented frequencies of laminated panels on elastic foundations using a sophisticated theory of hyperbolic shear strain. Taibi et al. [20] presented a thermo-mechanical analysis of functionally graded sandwich panels on elastic supports using the higher-order shear strain theory. Zaoui [21] developed 2D and 3D high-order shear strain models to calculate the frequencies of functionally graded panels on elastic supports. Atmane et al. [5] developed a new high-order shear strain model to calculate the frequencies of functionally graded panels on two parametric elastic substrates. Shinde et al. [19] developed a new higher-order shear deformation for the bending analysis of orthotropic plates under various loading conditions. Mahmoudi et al. [8] presented free vibrations analysis of a functionally graded plate resting on a Winkler-Pasternak elastic foundation using the Navier's technique based on a high-order shear deformation theory (HSDT). Fahsi et al. [6] developed a new refined quasi-3D shear deformation theory for bending, buckling, and free vibration analyses of a functionally graded porous beam resting on an elastic foundation involving three unknown functions. Naik and Sayyad [10] developed a fifth-order shear and normal deformation theory for the analysis of laminated composite and sandwich plates. Shen [18] presented a nonlinear bending analysis of a simply supported shear deformable cross-ply laminated plate with piezoelectric actuators subjected to a transverse uniform or sinusoidal load combined with electrical loads and in thermal environments.

Advantages of the proposed theory:

1. The present theory considers the effects of transverse normal strain, which is neglected in many theories available in the literature including the theory of Reddy [12].
2. The theory plots realistic distributions of the transverse shear stress through the plate thickness and satisfies the shear-free conditions at the extreme surfaces of the panel.
3. The present theory does not require a shearing correction factor to account for the strain energy caused by shear deformation.
4. The above literature review indicates that there is limited research on the effect of Winkler and Pasternak parameters on the behavior of laminated panels. Therefore, the present

high-order computational model, which accounts for the effects of both shearing and normal strains, is developed here to predict the bending behavior of simply supported cross-laminate panels resting on elastic 2-parameter foundations.

2. Materials and methods

2.1. Types of elastic foundation

The Winkler foundation is a parametric model that uses elastic springs and has a linear relationship between load and deflection. The Pasternak foundations are formed by connecting the ends of springs to plates or "shear layers" consisting of incompressible vertical elements that can be deformed only by lateral shear. In this study, the laminated panel is assumed to rest on a 2-parameter (Winkler and Pasternak) foundation with the Winkler spring stiffness K_0 and the shear stiffness K_1. The relationship between these stiffnesses and the lay-down of the laminated sheets is given by the reaction force F [13]

$$F = k_0 w - k_1 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) w, \tag{1}$$

where w is the plate deflection.

2.2. A laminated panel on an elastic foundation

Let us consider a laminated panel of length a, width b, and thickness h, as shown in Fig. 1. The laminate consists of N layers of linear elastic and orthotropic materials. The panel rests on a 2-parameter elastic foundation. A vertical load $q(x, y)$ is applied to the top surface of the panel, i.e., $z = -h/2$.

2.3. The displacement field

The displacement field of the present theory, considering the effects of the transverse normal strain, is of the following form [13]:

$$\begin{aligned} u(x, y, z) &= u_0(x, y) - z\frac{\partial w_0}{\partial x} + f(z)\phi(x, y) \ , \\ v(x, y, z) &= v_0(x, y) - z\frac{\partial w_0}{\partial y} + f(z)\psi(x, y) \ , \\ w(x, y, z) &= w_0(x, y) + g(z)\xi(x, y) \ , \end{aligned} \tag{2}$$

where the displacements of any point on the panel are u, v, w, the displacement of any point on the midplane are u_0, v_0, w_0, and the shear rotations are $\phi(x, y)$, $\psi(x, y)$, and $\xi(x, y)$. The function $f(z)$ is chosen based on the boundary conditions of the shearing strain. In this study, $f(z) = (h/\pi)\sin(\pi z/h)$ and $g(z) = \cos(\pi z/h)$.

2.4. Strain-displacement relationships

For the panel deformation, the strain quantities ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, γ_{yz} and the displacement quantities u, v, w follow the linear kinematic relations from the theory of elasticity [13]

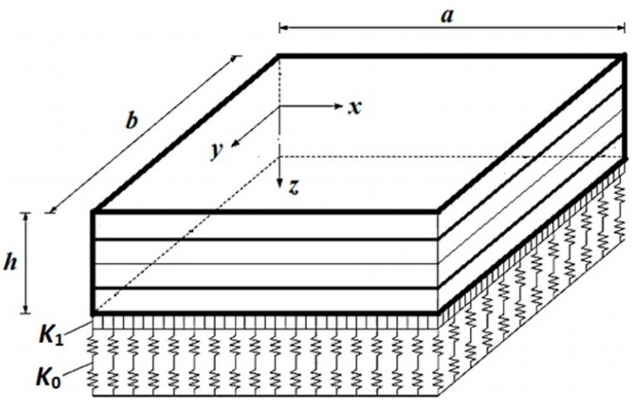

Fig. 1. The geometry and coordinate system of the laminated panel with an elastic foundation

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + f(z)\frac{\partial \phi}{\partial x}, \qquad & \varepsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + f(z)\frac{\partial \psi}{\partial y}, \\
\varepsilon_z &= g'(z)\xi\,, & \gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} + f(z)\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right), \\
\gamma_{xz} &= g(z)\left(\phi + \frac{\partial \xi}{\partial x}\right), & \gamma_{yz} &= g(z)\left(\psi + \frac{\partial \xi}{\partial y}\right).
\end{aligned} \tag{3}$$

2.5. *Stress-strain relationships*

The stress quantities of the k-th layer of the laminated panel are calculated using

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}_k, \tag{4}$$

where

$$\begin{aligned}
Q_{11} &= \frac{1-\mu_{23}\mu_{32}}{E_2E_3\Delta}, \qquad & Q_{12} &= \frac{\mu_{21}+\mu_{31}\mu_{23}}{E_2E_3\Delta}, \qquad & Q_{13} &= \frac{\mu_{31}+\mu_{21}\mu_{32}}{E_2E_3\Delta}, \\
Q_{22} &= \frac{1-\mu_{13}\mu_{31}}{E_1E_3\Delta}, & Q_{23} &= \frac{\mu_{32}+\mu_{12}\mu_{31}}{E_1E_3\Delta}, & Q_{33} &= \frac{1-\mu_{12}\mu_{21}}{E_1E_2\Delta}, \\
Q_{44} &= G_{23}, & Q_{55} &= G_{13}, & Q_{66} &= G_{12}, \\
\Delta &= 1-\mu_{12}\mu_{21}-\mu_{23}\mu_{32}-\mu_{31}\mu_{13}-2\mu_{21}\mu_{32}\mu_{13},
\end{aligned} \tag{5}$$

where Q_{ij} are the stiffness factors expressed by engineering constant, E_1, E_2, E_3 are the elastic moduli, μ_{12}, μ_{21}, μ_{23}, μ_{32}, μ_{13}, μ_{31} are the Poisson's ratios, and G_{12}, G_{23}, G_{13} are the shear moduli of the material.

2.6. Governing equations and boundary conditions

In the present study, we use the principle of virtual work to derive the governing equations and boundary conditions of the current theory [13]

$$\int_0^a \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy} \delta\gamma_{xy} + \tau_{xz} \delta\gamma_{xz} + \tau_{yz} \delta\gamma_{yz}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = $$

$$= \int_0^a \int_0^b q(x, y) \, \delta w \, \mathrm{d}x \, \mathrm{d}y. \tag{6}$$

Substituting all the strain quantities from (3) in (6) and integrating over the variable z, one can rewrite (6) as follows:

$$\int_0^a \int_0^b \left[N_x \frac{\partial}{\partial x} (\delta u_0) - M_x^c \frac{\partial^2}{\partial x^2} (\delta w_0) + M_x^s \frac{\partial}{\partial x} (\delta \phi) + N_y \frac{\partial}{\partial y} (\delta v_0) - M_y^c \frac{\partial^2}{\partial y^2} (\delta w_0) + \right.$$

$$\left. + M_y^s \frac{\partial}{\partial y} (\delta \psi) + Q_z \delta \xi \right] \mathrm{d}x \, \mathrm{d}y + \int_0^a \int_0^b \left\{ N_{xy} \left[\frac{\partial}{\partial y} (\delta u_0) + \frac{\partial}{\partial x} (\delta v_0) \right] - 2M_{xy}^c \frac{\partial^2}{\partial x \partial y} (\delta w_0) + \right.$$

$$\left. + M_y^s \left[\frac{\partial}{\partial y} (\delta \phi) + \frac{\partial}{\partial x} (\delta \psi) \right] + Q_x \left[\delta \phi + \frac{\partial}{\partial x} (\delta \xi) \right] + Q_y \left[\delta \psi + \frac{\partial}{\partial y} (\delta w_0) \right] \right\} \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_0^a \int_0^b \left(q(x, y) + k_0 w_0 - k_1 \frac{\partial^2 w_0}{\partial x^2} - k_1 \frac{\partial^2 w_0}{\partial y^2} \right) \delta w \, \mathrm{d}x \, \mathrm{d}y, \tag{7}$$

where the resultant in-plane stresses are denoted by N_x, N_y, N_{xy}, the resultant bending moments are denoted by M_x^c, M_y^c, M_{xy}^c, the resultant higher order moments are denoted by M_x^s, M_y^s, M_{xy}^s, and the resultant shear forces are denoted by Q_x, Q_y, Q_z. The aforementioned resultant quantities are defined as:

1) resultant in-plane stresses

$$\begin{aligned}
N_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, \mathrm{d}z = A_{11} \tfrac{\partial u_0}{\partial x} - B_{11} \tfrac{\partial^2 w_0}{\partial x^2} + C_{11} \tfrac{\partial \phi}{\partial x} + A_{12} \tfrac{\partial v_0}{\partial y} - B_{12} \tfrac{\partial^2 w_0}{\partial y^2} + C_{12} \tfrac{\partial \psi}{\partial y} + D_{13} \, \xi, \\
N_y &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, \mathrm{d}z = A_{12} \tfrac{\partial u_0}{\partial x} - B_{12} \tfrac{\partial^2 w_0}{\partial x^2} + C_{12} \tfrac{\partial \phi}{\partial x} + A_{22} \tfrac{\partial v_0}{\partial y} - B_{22} \tfrac{\partial^2 w_0}{\partial y^2} + C_{22} \tfrac{\partial \psi}{\partial y} + D_{23} \, \xi, \\
N_{xy} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \, \mathrm{d}z = A_{66} \tfrac{\partial u_0}{\partial y} + A_{66} \tfrac{\partial v_0}{\partial x} - 2B_{66} \tfrac{\partial^2 w_0}{\partial x \partial y} + C_{66} \left(\tfrac{\partial \phi}{\partial y} + \tfrac{\partial \psi}{\partial x} \right),
\end{aligned} \tag{8}$$

2) resultant bending moments

$$M_x^c = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z \, \mathrm{d}z = B_{11}\frac{\partial u_0}{\partial x} - F_{11}\frac{\partial^2 w_0}{\partial x^2} + H_{11}\frac{\partial \phi}{\partial x} + B_{12}\frac{\partial v_0}{\partial y} - F_{12}\frac{\partial^2 w_0}{\partial y^2} + H_{12}\frac{\partial \psi}{\partial y} + I_{13}\,\xi,$$

$$M_y^c = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z \, \mathrm{d}z = B_{12}\frac{\partial u_0}{\partial x} - F_{12}\frac{\partial^2 w_0}{\partial x^2} + H_{12}\frac{\partial \phi}{\partial x} + B_{22}\frac{\partial v_0}{\partial y} - F_{22}\frac{\partial^2 w_0}{\partial y^2} + H_{22}\frac{\partial \psi}{\partial y} + I_{23}\,\xi, \quad (9)$$

$$M_{xy}^c = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z \, \mathrm{d}z = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{66}\gamma_{xy} z \, \mathrm{d}z = B_{66}\frac{\partial u_0}{\partial y} + B_{66}\frac{\partial v_0}{\partial x} - 2F_{66}\frac{\partial^2 w_0}{\partial x \partial y} + H_{66}\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right),$$

3) resultant higher-order moments

$$M_x^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x f(z) \, \mathrm{d}z = C_{11}\frac{\partial u_0}{\partial x} - H_{11}\frac{\partial^2 w_0}{\partial x^2} + J_{11}\frac{\partial \phi}{\partial x} + C_{12}\frac{\partial v_0}{\partial y} - H_{12}\frac{\partial^2 w_0}{\partial y^2} + J_{12}\frac{\partial \psi}{\partial y} + L_{13}\,\xi,$$

$$M_y^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y f(z) \, \mathrm{d}z = C_{12}\frac{\partial u_0}{\partial x} - H_{12}\frac{\partial^2 w_0}{\partial x^2} + J_{12}\frac{\partial \phi}{\partial x} + C_{22}\frac{\partial v_0}{\partial y} - H_{22}\frac{\partial^2 w_0}{\partial y^2} + J_{22}\frac{\partial \psi}{\partial y} + L_{23}\,\xi,$$

$$M_{xy}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} f(z) \, \mathrm{d}z = C_{66}\frac{\partial u_0}{\partial y} + C_{66}\frac{\partial v_0}{\partial x} - 2H_{66}\frac{\partial^2 w_0}{\partial x \partial y} + J_{66}\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right), \quad (10)$$

4) resultant shear forces

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz}\, g(z) \, \mathrm{d}z = M_{55}\left(\phi + \frac{\partial \xi}{\partial x}\right),$$

$$Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz}\, g(z) \, \mathrm{d}z = M_{44}\left(\psi + \frac{\partial \xi}{\partial y}\right), \quad (11)$$

$$Q_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z \frac{\partial g(z)}{\partial z} \, \mathrm{d}z = D_{13}\frac{\partial u_0}{\partial x} - I_{13}\frac{\partial^2 w_0}{\partial x^2} + L_{13}\frac{\partial \phi}{\partial x} + D_{23}\frac{\partial v_0}{\partial y} - I_{23}\frac{\partial^2 w_0}{\partial y^2} + L_{23}\frac{\partial \psi}{\partial y} + S_{33}\,\xi,$$

where

$$\begin{aligned}
&A_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathrm{d}z, && B_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\, \mathrm{d}z, && C_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z)\, \mathrm{d}z,\\
&D_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} g'(z)\, \mathrm{d}z, && F_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\, \mathrm{d}z, && H_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} f(z) z\, \mathrm{d}z,\\
&I_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} z g'(z)\, \mathrm{d}z, && J_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[f(z)\right]^2\, \mathrm{d}z, && L_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} g'(z) f(z)\, \mathrm{d}z,\\
&M_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[g(z)\right]^2\, \mathrm{d}z, && S_{ij} = Q_{ij}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[g'(z)\right]^2\, \mathrm{d}z.
\end{aligned} \quad (12)$$

The integration of (7) after imposing the fundamental lemma of calculus yields the following equations of the present theory:

$$\begin{aligned}
\delta u_0: &\quad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0\ ,\\
\delta v_0: &\quad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0\ ,\\
\delta w_0: &\quad \frac{\partial^2 M_x^c}{\partial x^2} + 2\frac{\partial^2 M_{xy}^c}{\partial x \partial y} + \frac{\partial^2 M_y^c}{\partial y^2} + q + \left(k_0 w_0 - k_1 \frac{\partial^2 w_0}{\partial x^2} - k_1 \frac{\partial^2 w_0}{\partial y^2}\right) = 0\ ,\\
\delta \phi: &\quad \frac{\partial M_x^s}{\partial x} + \frac{\partial M_{xy}^s}{\partial x} - Q_x = 0\ ,\\
\delta \psi: &\quad \frac{\partial M_y^s}{\partial y} + \frac{\partial M_{xy}^s}{\partial x} - Q_y = 0\ ,\\
\delta \xi: &\quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial_y} - Q_z = 0.
\end{aligned} \tag{13}$$

The substitution of stress resultants from (8)–(12) into the set of governing equations (13), allows us to write the final governing equations associated with the unknowns in the present theory, i.e., δu_0, δv_0, δw_0, $\delta\phi$, $\delta\psi$, $\delta\xi$,

$$\begin{aligned}
\delta u_0: &A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_0}{\partial x^3} + C_{11}\frac{\partial^2 \phi}{\partial x^2} + A_{12}\frac{\partial^2 v_0}{\partial x \partial y} - B_{12}\frac{\partial^3 w_0}{\partial x \partial y^2} + C_{12}\frac{\partial^2 \psi}{\partial x \partial y} + D_{13}\frac{\partial \xi}{\partial x} +\\
&+ A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x \partial y} - 2B_{66}\frac{\partial^3 w_0}{\partial x \partial y^2} + C_{66}\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) = 0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\delta v_0: &A_{12}\frac{\partial^2 u_0}{\partial x \partial y} - B_{12}\frac{\partial^3 w_0}{\partial x^2 \partial y} + C_{12}\frac{\partial^2 \phi}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} - B_{22}\frac{\partial^3 w_0}{\partial y^3} + C_{22}\frac{\partial^2 \psi}{\partial y^2} + D_{23}\frac{\partial \xi}{\partial y} +\\
&+ A_{66}\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial^2 v_0}{\partial x^2} - 2B_{66}\frac{\partial^3 w_0}{\partial x^2 \partial y} + C_{66}\left(\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2}\right) = 0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\delta w_0: &B_{11}\frac{\partial^3 u_0}{\partial x^3} - F_{11}\frac{\partial^4 w_0}{\partial x^4} + H_{11}\frac{\partial^3 \phi}{\partial x^3} + B_{12}\frac{\partial^3 v_0}{\partial y \partial x^2} - F_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + H_{12}\frac{\partial^3 \psi}{\partial x^2 \partial y} + I_{13}\frac{\partial^2 \xi}{\partial x^2} +\\
&+ B_{12}\frac{\partial^3 u_0}{\partial x \partial y^2} - F_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + H_{12}\frac{\partial^3 \phi}{\partial x \partial y^2} + B_{22}\frac{\partial^3 v_0}{\partial y^3} - F_{22}\frac{\partial^4 w_0}{\partial y^4} + H_{22}\frac{\partial^3 \psi}{\partial y^3} +\\
&+ I_{23}\frac{\partial^2 \xi}{\partial y^2} + 2B_{66}\frac{\partial^3 u_0}{\partial x \partial y^2} + 2B_{66}\frac{\partial^3 v_0}{\partial x^2 \partial y} - 4F_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + 2H_{66}\left(\frac{\partial^3 \phi}{\partial x \partial y^2} + \frac{\partial^3 \psi}{\partial x^2 \partial y}\right) =\\
&= q - k_0 w_0 + k_1\frac{\partial^2 w_0}{\partial x^2} + k_1\frac{\partial^2 w_0}{\partial y^2},
\end{aligned} \tag{16}$$

$$\begin{aligned}
\delta \phi: &C_{11}\frac{\partial^2 u_0}{\partial x^2} - H_{11}\frac{\partial^3 w_0}{\partial x^3} + J_{11}\frac{\partial^2 \phi}{\partial x^2} + C_{12}\frac{\partial^2 v_0}{\partial x \partial y} - H_{12}\frac{\partial^3 w_0}{\partial x \partial y^2} + J_{12}\frac{\partial^2 \psi}{\partial x \partial y} + L_{13}\frac{\partial \xi}{\partial x} +\\
&+ C_{66}\frac{\partial^2 u_0}{\partial y^2} + C_{66}\frac{\partial^2 v_0}{\partial x \partial y} - 2H_{66}\frac{\partial^3 w_0}{\partial x \partial y^2} + J_{66}\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) - M_{55}\left(\phi + \frac{\partial \xi}{\partial x}\right) = 0,
\end{aligned} \tag{17}$$

$$\delta\psi : C_{12}\frac{\partial^2 u_0}{\partial x\partial y} - H_{12}\frac{\partial^3 w_0}{\partial x^2\partial y} + J_{12}\frac{\partial^2 \phi}{\partial x\partial y} + C_{22}\frac{\partial^2 v_0}{\partial y^2} - H_{22}\frac{\partial^3 w_0}{\partial y^3} + J_{22}\frac{\partial^2 \psi}{\partial y^2} + L_{23}\frac{\partial \xi}{\partial y} + \\ + C_{66}\frac{\partial^2 u_0}{\partial x\partial y} + C_{66}\frac{\partial^2 v_0}{\partial x^2} - 2H_{66}\frac{\partial^3 w_0}{\partial x^2\partial y} + J_{66}\left(\frac{\partial^2 \phi}{\partial x\partial y} + \frac{\partial^2 \psi}{\partial x^2}\right) - M_{44}\left(\psi + \frac{\partial \xi}{\partial y}\right) = 0, \tag{18}$$

$$\delta\xi : M_{55}\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 \xi}{\partial x^2}\right) + M_{44}\left(\frac{\partial \psi}{\partial y} + \frac{\partial^2 \xi}{\partial y^2}\right) - D_{13}\frac{\partial u_0}{\partial x} + I_{13}\frac{\partial^2 w_0}{\partial x^2} - L_{13}\frac{\partial \phi}{\partial x} - D_{23}\frac{\partial v_0}{\partial y} + \\ + I_{23}\frac{\partial^2 w_0}{\partial y^2} - L_{23}\frac{\partial \psi}{\partial y} - S_{33}\,\xi = 0. \tag{19}$$

3. Analytical solutions

In this section, we consider proving the validity and accuracy of the current theory for predicting the bending response of simply supported laminated panels on 2-parameter elastic supports. The panel is subjected to vertical load $q(x,y)$ on the top surface (i.e., $z = -h/2$).

The Navier's solution, which satisfies the simply supported boundary condition exactly, is commonly used for local and global solutions of the simply supported boundary condition of the plate. The simply supported boundary condition for the panel is

$$\begin{aligned} &\text{at } x = 0,\ x = a: \quad w = \psi = \xi = M_x = M_x^s = 0, \\ &\text{at } y = 0,\ y = b: \quad w = \phi = \xi = M_y = M_y^s = 0. \end{aligned} \tag{20}$$

Navier suggested taking the unknown variables in the following double trigonometric form to get the value given by (20) exactly [13]

$$\begin{aligned} u_0 &= u_{mn}\cos(\alpha x)\sin(\beta y), & v_0 &= v_{mn}\sin(\alpha x)\cos(\beta y), \\ w_0 &= w_{mn}\sin(\alpha x)\sin(\beta y), & \phi &= \phi_{mn}\cos(\alpha x)\sin(\beta y), \\ \psi &= \psi_{mn}\sin(\alpha x)\cos(\beta y), & \xi &= \xi_{mn}\sin(\alpha x)\sin(\beta y), \end{aligned} \tag{21}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$, u_{mn}, v_{mn}, w_{mn}, ϕ_{mn}, ψ_{mn}, ξ_{mn} are the unknowns to be determined, and m, n are the positive integers ranging from 1 to ∞. Using the Navier's solution method, the vertical load can be expressed in double trigonometric series [13]

$$q(x,y) = \sum_{m=1,3,5}^{\infty}\ \sum_{n=1,3,5}^{\infty} q_{mn}\sin(\alpha x)\ \sin(\beta y), \tag{22}$$

where q_{mn} is the factor of the Fourier expansion of the vertical load [13]. The value of this factor for sinusoidal and uniformly distributed loads is

$$\begin{aligned} &\text{sinusoidal loading } (m = 1,\ n = 1): & q_{mn} &= q_0, \\ &\text{uniform loading } (m = n = 1, 3, 5\ldots,\infty): & q_{mn} &= \frac{16q_0}{mn\pi^2}. \end{aligned} \tag{23}$$

The solution to the elastically supported laminate bending problem is obtained after inserting (21) and (22) into (14)–(19)

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \\ \phi_{mn} \\ \psi_{mn} \\ \xi_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{mn} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

where K_{ij} are the stiffness coefficients of the panel given as

$$\begin{aligned}
&K_{11} = -\left(-A_{11}\alpha^2 + A_{66}\beta^2\right), && K_{12} = -\left(A_{12}\alpha\beta + A_{66}\alpha\beta\right), \\
&K_{13} = B_{11}\alpha^3 + B_{12}\alpha\beta^2 + 2B_{66}\alpha\beta^2, && K_{14} = -\left(C_{11}\alpha^2 + C_{66}\beta^2\right), \\
&K_{15} = -\left(C_{12}\alpha\beta + C_{66}\alpha\beta\right), && K_{16} = D_{13}\alpha, \\
&K_{22} = -\left(A_{66}\alpha^2 - A_{22}\beta^2\right), && K_{23} = B_{12}\alpha^2\beta + B_{22}\beta^3 + 2B_{66}\alpha^2\beta, \\
&K_{24} = -\left(C_{12}\alpha\beta + C_{66}\alpha\beta\right), && K_{25} = -\left(C_{66}\alpha^2 + C_{22}\beta^2\right), \\
&K_{26} = D_{23}\beta, && \\
&K_{33} = -\left(F_{11}\alpha^4 + 2F_{12}\alpha^2\beta^2 + F_{22}\beta^4 + 4F_{66}\alpha^2\beta^2 + k_0 + k_1\alpha^2 + k_1\beta^2\right), && \\
&K_{34} = H_{11}\alpha^3 + H_{12}\alpha\beta^2 + 2H_{66}\alpha\beta^2, && K_{35} = H_{22}\beta^3 + 2H_{66}\alpha^2\beta + H_{12}\alpha^2\beta, \\
&K_{36} = -\left(I_{13}\alpha^2 + I_{23}\beta^2\right), && K_{44} = -\left(J_{11}\alpha^2 + J_{66}\beta^2 + M_{55}\right), \\
&K_{45} = -\left(J_{12}\alpha\beta + J_{66}\alpha\beta\right), && K_{46} = L_{13}\alpha - M_{55}\alpha, \\
&K_{55} = -\left(J_{22}\beta^2 + J_{66}\alpha^2 + M_{44}\right), && K_{56} = L_{23}\beta - M_{44}\beta, \\
&K_{66} = -\left(M_{55}\alpha^2 + M_{44}\beta^2 + S_{33}\right). &&
\end{aligned} \tag{25}$$

It is a well-known property of the stiffness matrix that it is always a symmetric matrix, i.e., $K_{ij} = K_{ji}$. The solution of (24) indicates the value of the unknown contained in the trigonometric form of the variable. Using these values, equations (2)–(5) can be used to calculate the displacement and stresses and their distributions in the thickness direction. Estimating the interlaminar transverse shear stress, we find that the constitutive relationship predicts two values of transverse shear stress at the ply interface due to changes in material properties. This phenomenon is unacceptable for the design of panel structures. Therefore, in this study, the transverse shear stress is determined using the equilibrium equation of the elastic theory. The equilibrium equations for the elastic theory are

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0, \quad \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z} = 0, \quad \frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0. \tag{26}$$

4. Illustrative examples

To validate the current theory, the following problems are solved and presented:

1. Bending analysis of a 2-layered ($0°/90°$) laminated panel supported on an elastic foundation under sinusoidal loading.
2. Bending analysis of 2-layered ($0°/90°$) laminated composite panel supported on elastic foundation under uniform loading.
3. Bending analysis of 3-layered ($0°/90°/0°$) laminated composite panel supported on elastic foundation under sinusoidal loading.

4. Bending analysis of 3-layered $(0°/90°/0°)$ laminated composite panel supported on elastic foundation under uniform loading.

Numerical results are shown in Tables 1–4 and graphical results are shown in Figs. 2 and 3. The following material properties are used to obtain numerical results corresponding to the bending behavior of the laminated panel resting on elastic foundation:

$$\frac{E_1}{E_2} = 25, \qquad E_3 = E_2, \qquad G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2, \qquad \mu_{12} = \mu_{31} = \mu_{23} = 0.25, \qquad \frac{\mu_{ij}}{\mu_{ji}} = \frac{E_i}{E_j}. \tag{27}$$

For the comparison purpose, the numerical results are presented in the following non-dimensional form:

$$\bar{w}(x,y,z) = \frac{100E_2}{q_0 h S^4}\, w\left(\frac{a}{2},\frac{b}{2},0\right), \qquad (\bar{\sigma}_x,\bar{\sigma}_y)(x,y,z) = \frac{1}{q_0 S^2}\,(\sigma_x,\sigma_y)\left(\frac{a}{2},\frac{b}{2},\pm\frac{h}{2}\right),$$
$$\bar{\tau}_{xy}(x,y,z) = \frac{1}{q_0 S^2}\,\tau_{xy}\left(0,0,\pm\frac{h}{2}\right), \qquad \bar{\tau}_{xz}(x,y,z) = \frac{1}{q_0 S}\,\tau_{xz}\left(0,\frac{b}{2},0\right), \tag{28}$$
$$\bar{\tau}_{yz}(x,y,z) = \frac{1}{q_0 S}\,\tau_{yz}\left(\frac{a}{2},0,0\right), \qquad S = \frac{a}{h}, \qquad K_0 = \frac{k_0 b^4}{E_2 h^3}, \qquad K_1 = \frac{k_1 b^2}{E_2 h^3}.$$

4.1. Discussion on numerical results

Problem 1: Displacements and stresses for 2-layered $(0°/90°)$ laminated composite panels supported by the 2-parameter elastic foundation under sinusoidal loading are summarized in Table 1. Both layers have the same thickness $(h/2)$ and have the material properties given by (27). The dimensionless forms of displacement and stress are given by (1). It can be seen from Table 1 that the displacements and stresses obtained using the current theory and the Reddy's model [12] are in very good agreement, whereas the Mindlin's model [9] does not account for the effect of the transverse normal strain. The classical plate theory (CPT) underestimates these results by ignoring both transverse shear and normal strains. The displacement and stress values decrease as the foundation stiffness and panel thickness increase. An increase in the aspect ratio $(S = a/h)$ causes a decrease in the values of displacements and stresses. The effects of foundation modulus on the non-dimensional values of displacements and stresses for 2-layered laminated panels under sinusoidal load with an aspect ratio of 4 are shown in Fig. 2. This figure reveals that the values of non-dimensional displacements and stresses are predicted more when the panel is not resting on elastic foundation. The elastic foundation decreases the values of displacements and stresses.

Problem 2: This problem applies the current theory to the bending analysis of a 2-layered $(0°/90°)$ laminated composite panel placed on a two-parametric elastic foundation and subjected to uniform loading. The material properties of the panel and the dimensionless form of the numerical results are similar to Problem 1. Numerical results are shown in Table 2, and the distribution of displacement and stress in the thickness direction is shown in Fig. 3. Table 2 clearly shows that the current theory agrees very well with the Reddy's model [12]. Also, FSDT and CPT are not accurate enough to capture the bending response of laminated panels on elastic substrates. This is because the shear and normal strain effects are ignored. Table 3 also reveals that the values of non-dimensional displacements and stresses are predicted more in the case of uniform load as compared to sinusoidal loadings. The values of displacements and stresses are

Table 1. Comparison of numerical results for 2-layered $(0°/90°)$ laminated panels placed on 2-parameter elastic foundations under sinusoidal loads

S	K_0	K_1	theory	w	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	0	0	present	1.9547	0.9064	0.9064	0.0562	0.3370	0.3370
			ref. [12]	1.9985	0.9060	0.9060	0.0577	0.3396	0.3396
			ref. [9]	1.9682	0.7157	0.7157	0.0525	0.3356	0.3356
			CPT	1.0636	0.7157	0.7157	0.0525	0.3356	0.3356
	100	0	present	0.6574	0.3067	0.3067	0.0190	0.1141	0.1141
			ref. [12]	0.6665	0.3022	0.3022	0.0192	0.1132	0.1132
			ref. [9]	0.6631	0.2273	0.2273	0.0167	0.1066	0.1066
			CPT	0.5154	0.3468	0.3468	0.0254	0.1626	0.1626
	100	10	present	0.2851	0.1330	0.1330	0.0082	0.0495	0.0495
			ref. [12]	0.2878	0.1305	0.1305	0.0083	0.0489	0.0489
			ref. [9]	0.2872	0.0968	0.0968	0.0071	0.0454	0.0454
			CPT	0.2555	0.1719	0.1719	0.0126	0.0806	0.0806
10	0	0	present	1.2089	0.7471	0.7471	0.0530	0.3352	0.3352
			ref. [12]	1.2161	0.7468	0.7468	0.0533	0.3357	0.3357
			ref. [9]	1.2083	0.7157	0.7157	0.0525	0.3356	0.3356
			CPT	1.0636	0.7157	0.7157	0.0525	0.3356	0.3356
	100	0	present	0.5487	0.3391	0.3391	0.0241	0.1521	0.1521
			ref. [12]	0.5488	0.3370	0.3370	0.0241	0.1515	0.1515
			ref. [9]	0.5472	0.3199	0.3199	0.0235	0.1500	0.1500
			CPT	0.5154	0.3468	0.3468	0.0254	0.1626	0.1626
	100	10	present	0.2640	0.1632	0.1632	0.0116	0.0732	0.0732
			ref. [12]	0.2634	0.1618	0.1618	0.0116	0.0727	0.0727
			ref. [9]	0.2631	0.1530	0.1530	0.0112	0.0717	0.0717
			CPT	0.2555	0.1719	0.1719	0.0126	0.0806	0.0806
100	0	0	present	1.0644	0.7177	0.7177	0.0525	0.3356	0.3356
			ref. [12]	1.0650	0.7160	0.7160	0.0525	0.3356	0.3356
			ref. [9]	1.0651	0.7157	0.7157	0.0525	0.3356	0.3356
			CPT	1.0636	0.7157	0.7157	0.0525	0.3356	0.3356
	100	0	present	0.5156	0.3477	0.3477	0.0254	0.1626	0.1626
			ref. [12]	0.5158	0.3467	0.3467	0.0254	0.1625	0.1625
			ref. [9]	0.5157	0.3466	0.3466	0.0254	0.1625	0.1625
			CPT	0.5154	0.3468	0.3468	0.0254	0.1626	0.1626
	100	10	present	0.2555	0.1723	0.1723	0.0126	0.0806	0.0806
			ref. [12]	0.2556	0.1718	0.1718	0.0126	0.0805	0.0805
			ref. [9]	0.2556	0.1717	0.1717	0.0126	0.0805	0.0805
			CPT	0.2555	0.1719	0.1719	0.0126	0.0806	0.0806

observed lower when the elastic foundation is provided and higher when the elastic foundation is not provided. This is also presented in Fig. 3.

Problem 3: In this problem, a 3-layered laminated panel is analyzed. The effects of the 2-parameter elastic supports on the dimensionless displacement and stress of laminated panels

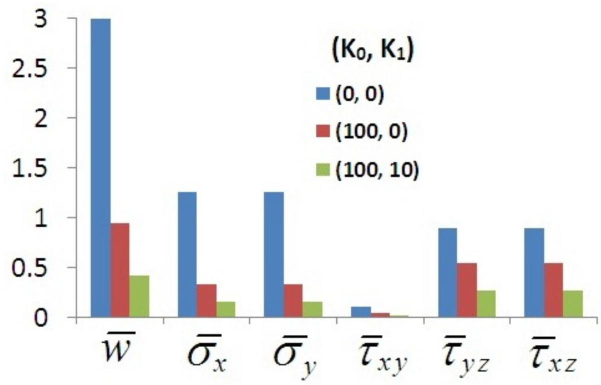

Fig. 2. Effects of foundation modulus on the non-dimensional values of displacements and stresses for 2-layered $(0°/90°)$ laminated panels under uniform load with an aspect ratio of 4

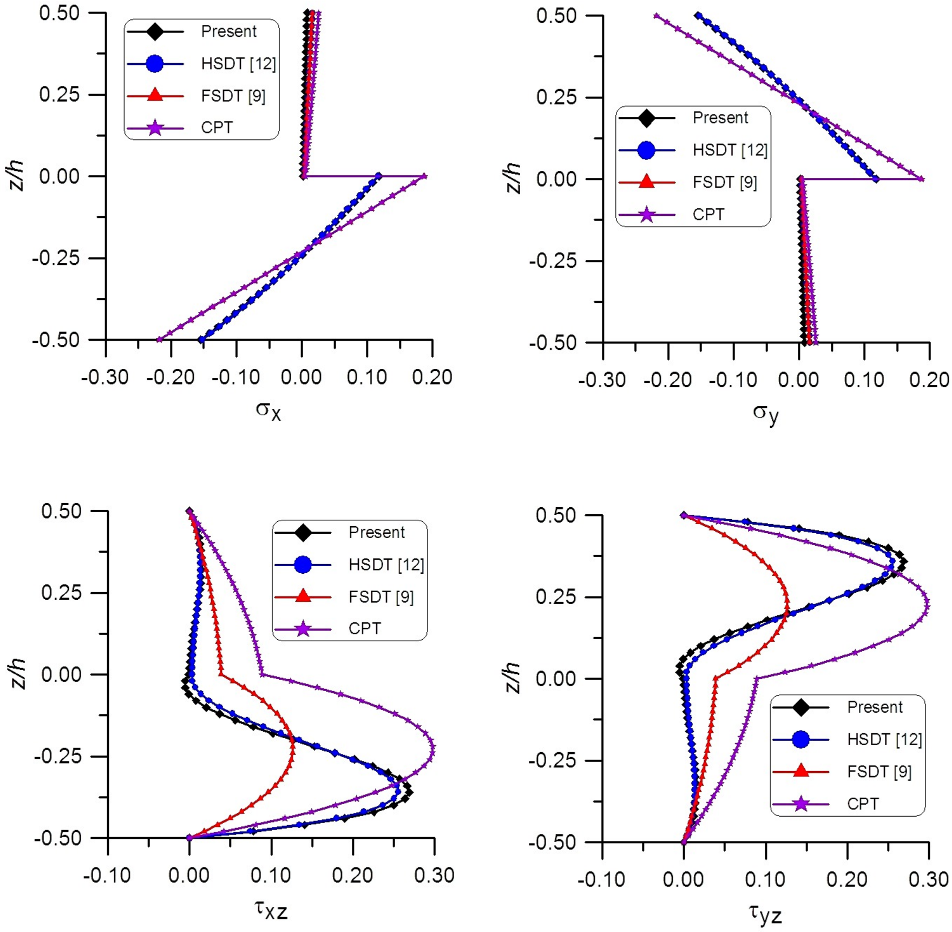

Fig. 3. Through-thickness displacement and stress distribution of a 2-layered $(0°/90°)$ laminated composite panel resting on elastic foundation under uniformly distributed load ($a/h = 4$, $K_0 = 100$, $K_1 = 10$)

with different a/h ratios are investigated in Table 3. The results developed using the current theory are compared with the Reddy's model [12], Mindlin's model [9], and CPT. This comparison demonstrates the validity and accuracy of the current theory. Table 3 shows that the present results are in good agreement with those presented by Reddy [12] and Mindlin [9]. CPT and

Table 2. Comparison of numerical results for 2-layered $(0°/90°)$ laminated panels placed on 2-parameter elastic foundations under uniform load

S	K_0	K_1	theory	w	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	0	0	present	2.9983	1.2603	1.2603	0.1104	0.8945	0.8945
			ref. [12]	3.0796	1.2691	1.2691	0.1070	0.8648	0.8648
			ref. [9]	3.0082	1.0636	1.0636	0.0992	0.7265	0.7265
			CPT	1.6957	1.0763	1.0763	0.0934	0.7415	0.7415
	100	0	present	0.9433	0.3330	0.3330	0.0481	0.5427	0.5427
			ref. [12]	0.9420	0.3305	0.3305	0.0430	0.5068	0.5068
			ref. [9]	0.9288	0.2938	0.2938	0.0391	0.3327	0.3327
			CPT	0.8077	0.4813	0.4813	0.0494	0.4577	0.4577
	100	10	present	0.4167	0.1550	0.1550	0.0207	0.2688	0.2688
			ref. [12]	0.4141	0.1521	0.1521	0.0182	0.2554	0.2554
			ref. [9]	0.4120	0.2179	0.2179	0.0155	0.1262	0.1262
			CPT	0.3929	0.2181	0.2181	0.0271	0.2974	0.2974
10	0	0	present	1.9070	1.1057	1.1057	0.0978	0.7545	0.7545
			ref. [12]	1.9173	1.1049	1.1049	0.0977	0.7530	0.7530
			ref. [9]	1.9049	1.0717	1.0717	0.0961	0.7369	0.7369
			CPT	1.6955	1.0763	1.0763	0.0934	0.7415	0.7415
	100	0	present	0.8400	0.4515	0.4515	0.0504	0.4632	0.4632
			ref. [12]	0.8389	0.4480	0.4480	0.0498	0.4557	0.4557
			ref. [9]	0.8366	0.4365	0.4365	0.0486	0.4288	0.4288
			CPT	0.8077	0.4813	0.4813	0.0494	0.4577	0.4577
	100	10	present	0.3966	0.2033	0.2033	0.0267	0.2812	0.2812
			ref. [12]	0.3952	0.2015	0.2015	0.0260	0.2746	0.2746
			ref. [9]	0.3947	0.1982	0.1982	0.0251	0.2405	0.2405
			CPT	0.3929	0.2181	0.2181	0.0271	0.2974	0.2974
100	0	0	present	1.6967	1.0796	1.0796	0.0933	0.7406	0.7406
			ref. [12]	1.6976	1.0766	1.0766	0.0953	0.7414	0.7414
			ref. [9]	1.6976	1.0763	1.0763	0.0934	0.7415	0.7415
			CPT	1.6955	1.0763	1.0763	0.0934	0.7415	0.7415
	100	0	present	0.8080	0.4827	0.4827	0.0493	0.4567	0.4567
			ref. [12]	0.8081	0.4809	0.4809	0.0512	0.4574	0.4574
			ref. [9]	0.8080	0.4808	0.4808	0.0494	0.4574	0.4574
			CPT	0.8077	0.4813	0.4813	0.0494	0.4577	0.4577
	100	10	present	0.3930	0.2188	0.2188	0.0271	0.2963	0.2963
			ref. [12]	0.3929	0.2179	0.2179	0.0287	0.2967	0.2967
			ref. [9]	0.3929	0.2179	0.2179	0.0272	0.2967	0.2967
			CPT	0.3929	0.2181	0.2181	0.0271	0.2974	0.2974

FSDT underestimate the values of displacements and stresses due to neglecting the effects of transverse shear and normal strains. An increase in the layers of laminated panels decreases the values of displacements and stresses. Fig. 4 shows the effects of foundation modulus on the non-dimensional values of displacements and stresses for 3-layered laminated panels under

Table 3. Comparison of numerical results for 3-layered $(0°/90°/0°)$ laminated panels placed on 2-parameter elastic foundations under sinusoidal load

S	K_0	K_1	theory	w	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	0	0	present	1.9236	0.7535	0.0880	0.0496	0.2088	0.2768
			ref. [12]	1.9218	0.7345	0.0782	0.0497	0.2086	0.2855
			ref. [9]	1.5681	0.4370	0.0614	0.0369	0.1968	0.3368
			CPT	0.4312	0.5387	0.0267	0.0213	0.0823	0.3951
	100	0	present	0.6504	0.2577	0.0301	0.0170	0.0714	0.0947
			ref. [12]	0.6577	0.2514	0.0268	0.0170	0.0714	0.0977
			ref. [9]	0.6106	0.1574	0.0221	0.0133	0.0709	0.1213
			CPT	0.3013	0.3764	0.0186	0.0149	0.0575	0.2761
	100	10	present	0.2829	0.1121	0.0131	0.0074	0.0311	0.0412
			ref. [12]	0.2862	0.1094	0.0116	0.0074	0.0311	0.0425
			ref. [9]	0.2769	0.0696	0.0098	0.0059	0.0313	0.0536
			CPT	0.1889	0.2360	0.0117	0.0093	0.0361	0.1731
10	0	0	present	0.7154	0.5720	0.0411	0.0278	0.1179	0.3670
			ref. [12]	0.7125	0.5684	0.0387	0.0277	0.1167	0.3693
			ref. [9]	0.6306	0.5134	0.0353	0.0252	0.1108	0.3806
			CPT	0.4312	0.5387	0.0267	0.0213	0.0823	0.3951
	100	0	present	0.4176	0.3339	0.0240	0.0162	0.0688	0.2142
			ref. [12]	0.4161	0.3319	0.0226	0.0162	0.0682	0.2156
			ref. [9]	0.3867	0.3076	0.0211	0.0151	0.0664	0.2280
			CPT	0.3013	0.3764	0.0186	0.0149	0.0575	0.2761
	100	10	present	0.2292	0.1833	0.0132	0.0089	0.0378	0.1176
			ref. [12]	0.2284	0.1822	0.0124	0.0089	0.0374	0.1184
			ref. [9]	0.2193	0.1717	0.0118	0.0084	0.0371	0.1273
			CPT	0.1889	0.2360	0.0117	0.0093	0.0361	0.1731
100	0	0	present	0.4342	0.5397	0.0274	0.0213	0.0826	0.3947
			ref. [12]	0.4342	0.5390	0.0268	0.0214	0.0827	0.3948
			ref. [9]	0.4333	0.5384	0.0268	0.0213	0.0827	0.3950
			CPT	0.4312	0.5387	0.0267	0.0213	0.0823	0.3951
	100	0	present	0.3027	0.3763	0.0191	0.0149	0.0576	0.2752
			ref. [12]	0.3027	0.3758	0.0187	0.0149	0.0577	0.2753
			ref. [9]	0.3023	0.3755	0.0187	0.0149	0.0577	0.2755
			CPT	0.3013	0.3764	0.0186	0.0149	0.0575	0.2761
	100	10	present	0.1895	0.2356	0.0120	0.0093	0.0361	0.1723
			ref. [12]	0.1895	0.2352	0.0117	0.0093	0.0361	0.1723
			ref. [9]	0.1893	0.2351	0.0117	0.0093	0.0361	0.1725
			CPT	0.1889	0.2360	0.0117	0.0093	0.0361	0.1731

sinusoidal load with an aspect ratio of 4.

Problem 4: In this problem, the effects of uniform loading on the deflection of 3-layered laminated panels are investigated. The effects of base stiffness and a/h ratio on dimensionless displacement and stress are examined in Table 4. This comparison of the present results with

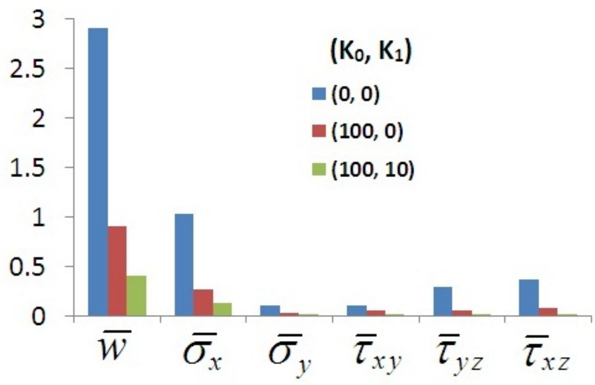

Fig. 4. Effects of foundation modulus on the non-dimensional values of displacements and stresses for 3-layered $(0°/90°/0°)$ laminated panels under uniform load with an aspect ratio of 4

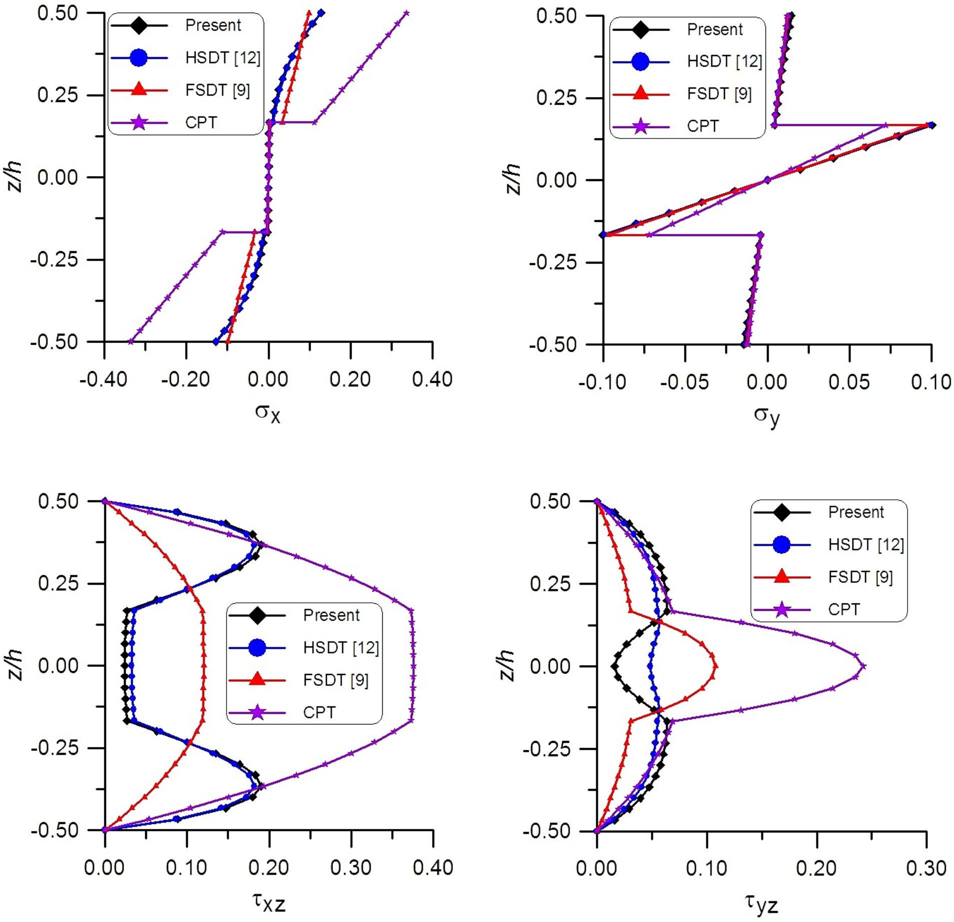

Fig. 5. Through-thickness displacement and stress distribution of a 3-layered $(0°/90°/0°)$ laminated composite panel resting on elastic foundation under uniformly distributed load ($a/h = 4$, $K_0 = 100$, $K_1 = 10$)

those obtained with HSDT, FSDT, and CPT clearly shows the effect of lateral vertical loading. Table 4 reveals that the present theory shows good agreement with the theory of Reddy [12]. Fig. 5 shows the through-the-thickness distributions of stresses for 3-layered laminated panels under uniform load with an aspect ratio of 4.

Table 4. Comparison of numerical results for 3-layered $(0°/90°/0°)$ laminated panels placed on 2-parameter elastic foundations under uniform load

S	K_0	K_1	theory	w	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{yz}$	$\bar{\tau}_{xz}$
4	0	0	present	2.9040	1.0343	0.1138	0.1097	0.2933	0.3751
			ref. [12]	2.9091	1.0177	0.1030	0.1092	0.3947	0.4036
			ref. [9]	2.3538	0.6546	0.0852	0.0736	0.5528	0.6395
			CPT	0.6660	0.8076	0.0307	0.0426	0.3859	0.7233
	100	0	present	0.9086	0.2711	0.0279	0.0533	0.0531	0.0850
			ref. [12]	0.9059	0.2705	0.0260	0.0529	0.1484	0.1030
			ref. [9]	0.8412	0.2137	0.0253	0.0333	0.3133	0.2838
			CPT	0.4573	0.5472	0.0185	0.0319	0.3374	0.5324
	100	10	present	0.4097	0.1286	0.0146	0.0217	0.0161	0.0241
			ref. [12]	0.4090	0.1269	0.0134	0.0222	0.0486	0.0329
			ref. [9]	0.3954	0.0986	0.0125	0.0134	0.1078	0.1203
			CPT	0.2861	0.3356	0.0118	0.0209	0.2421	0.3754
10	0	0	present	1.0954	0.8436	0.0510	0.0594	0.3553	0.6139
			ref. [12]	1.0900	0.8395	0.0481	0.0593	0.3859	0.6259
			ref. [9]	0.9642	0.7720	0.0442	0.0515	0.4230	0.7054
			CPT	0.6660	0.8076	0.0307	0.0426	0.3859	0.7233
	100	0	present	0.6181	0.4638	0.0249	0.0400	0.2633	0.3668
			ref. [12]	0.6151	0.4622	0.0235	0.0400	0.2942	0.3775
			ref. [9]	0.5736	0.4429	0.0227	0.0346	0.3374	0.4585
			CPT	0.4573	0.3356	0.0185	0.0319	0.3374	0.5324
	100	10	present	0.3381	0.2487	0.0141	0.0233	0.1432	0.2080
			ref. [12]	0.3367	0.2482	0.0134	0.0234	0.1642	0.2166
			ref. [9]	0.3245	0.2443	0.0132	0.0200	0.1937	0.2805
			CPT	0.2861	0.5472	0.0118	0.0209	0.2421	0.3754
100	0	0	present	0.6704	0.8089	0.0317	0.0428	0.3852	0.7211
			ref. [12]	0.6705	0.8080	0.0309	0.0428	0.3858	0.7216
			ref. [9]	0.6691	0.8073	0.0308	0.0427	0.3863	0.7231
			CPT	0.6660	0.8076	0.0307	0.0426	0.3859	0.7233
	100	0	present	0.4593	0.5468	0.0191	0.0320	0.3363	0.5294
			ref. [12]	0.4593	0.5462	0.0186	0.0320	0.3369	0.5298
			ref. [9]	0.4587	0.5460	0.0186	0.0319	0.3374	0.5314
			CPT	0.4573	0.5472	0.0185	0.0319	0.3374	0.5324
	100	10	present	0.2868	0.3348	0.0122	0.0210	0.2405	0.3721
			ref. [12]	0.2868	0.3344	0.0119	0.0210	0.2409	0.3724
			ref. [9]	0.2866	0.3344	0.0119	0.0209	0.2414	0.3741
			CPT	0.2861	0.3356	0.0118	0.0209	0.2421	0.3754

5. Conclusions

In this work, the sine and cosine theory was used for the bending analysis of simply supported layered composite panels on a 2-parameter elastic foundation, considering transverse normal strains. The analysis considered Winkler and Pasternak type elastic foundation and used the Navier method to develop an analytical solution. The authors investigated the effects of foundation stiffness, stacking sequence, and *a/h* ratio on the bending behavior of laminated panels. Comparative studies with existing theories were conducted to verify the validity and effectiveness of current theories. It can be concluded that the present theory is capable of predicting good results for thick laminated panels resting on an elastic foundation due to consideration of the effects of transverse normal strain. The displacement and stress values decreased as the foundation stiffness and panel thickness increased. The provision of an elastic foundation decreased the values of displacements and stresses. The values of non-dimensional displacements and stresses were predicted more in the case of uniform load as compared to sinusoidal loadings. In the future, this theory can be extended for the analysis of sandwich plates resting on an elastic foundation, the free vibration response of laminated and sandwich panels supported by an elastic foundation, the effects of angle-ply laminates, etc.

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