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Oscillatory convection in viscoelastic ferrofluid layer: Linear and non-linear stability analyses

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Abstract

The problem of ferroconvection in a viscoelastic fluid layer is studied with the aim to investigate oscillatory motions. In this article, a stability analysis for both linear and nonlinear systems is carried out. In the linear stability analysis, the expressions for steady and oscillatory Rayleigh numbers are obtained and the effects of magnetic as well as viscoelastic parameters on the onset of viscoelastic ferromagnetic convection are investigated numerically. From the analysis, we found that the magnetic number (M_1), the stress relaxation time (λ_1) and the nonlinearity of magnetization (M_3) have destabilizing influences on the onset of ferroconvection, whereas the strain retardation time (λ_2) has stabilizing influence. In the weakly nonlinear stability analysis, the formula for heat transfer rate in terms of the Nusselt number is derived for oscillatory convection. From the analyses, we found that for increasing values of the magnetic number, stress relaxation time and nonlinearity of magnetization, the heat transfer rate rises, whereas it decreases for larger values of the strain retardation time. Moreover, the pitchfork bifurcation analysis yields that in order to reach the stable positions, the value of amplitude increases as the stress relaxation time increases, whereas a reverse trend is observed for the strain retardation time.

Keywords: non-linear stability, oscillatory convection, ferrofluids, viscoelastic, pitchfork bifurcation

1. Introduction

Stable colloidal synthetic mixtures of nanoscopic magnetic particles suspended in carrier liquids, such as kerosene, water, heptane or synthetic oil, are termed ferrofluids. These suspended particles (having a certain magnetic moment) are coated with some surfactants to prevent coagulation. Magnetic drug targeting hyperthermia, contrast intensification of magnetic resonance imaging (MRI), cooling of loudspeakers, sealing of rotating shafts, pressure seals of blowers and compressors, etc. are some of the important applications of ferrofluids in the fields of technology and biomedical sciences.

Currently, the study of viscoelastic ferromagnetic fluids is important, particularly the gel-based weakly electrically conducting fluids with micron-sized ferrite suspended particles, due to their variety of biomedical applications. Ferrofluids can be distinguished from other fluids primarily by their body coupling and polarisation forces. In the past few years, the study of the convective motions in ferrofluids has attracted considerable attention from researchers. A thorough description of ferrofluids and their exciting properties was presented by Rosensweig [23]. He found that the magnetic field, fluid density and fluid temperature change with varying magnetization, and thus, affect the heat transfer rate and the distribution of the fluid's body force. In recent years, substantial attention has been paid to investigate the dynamical behavior of ferrofluids under varying hydrodynamic assumptions.

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Ferroconvection (or ferrohydrodynamics) is the study of convective motions in ferrofluids under an applied magnetic field and is similar to standard thermal convection, see [6]. One of the striking features of ferrohydrodynamics is that the flow occurs even in the absence of Lorentz forces. Finlayson in [12] studied "the linear convective instability in a layer of ferromagnetic fluid heated from below permitted with a vertical magnetic field" using normal mode analysis for infinite amplitude disturbances. He showed from the eigenvalue problem that the overstable motions of increasing amplitude are prohibited under the assumption of extremely small values of the magnetic parameter ($M_2 \cong 0$), and thus, only the stationary mode of instability is possible. However, for non-zero values of the magnetic parameter M_2, Dhiman and Sharma in [9] proved that the overstable motions of rising amplitude can still exist under certain prescribed conditions. In [24], Siddheshwar studied "the oscillatory convection in a viscoelastic-Boussinesq-ferromagnetic fluid considering infinite magnetic susceptibility" and discussed the linear stability analysis. He claimed that the strain retardation time has stabilizing effects, whereas the stress relaxation time has destabilizing effects. Siddheshwar in [25] investigated different types of ferromagnetic viscoelastic models and discussed the stabilizing effect of the strain retardation time and destabilizing effect of the stress relaxation time and magnetization on the onset of the linear oscillatory instability in ferroconvection. In [14], Kanchana et al. studied cellular-convective and chaotic motions in a ferrofluid, revealing the advanced onset of regular convection and delayed chaotic motions, with vigorous-chaotic motions observed with vertical modes and also discussed the horizontal modes and multiple modes. Further, Dhiman and Sharma in [8] investigated the effect of temperature-dependent viscosity and boundary conditions on the ferroconvection problem and compared the values of critical Rayleigh numbers for different cases of boundary conditions.

Fluids are termed as non-Newtonian fluids when there is a nonlinear relationship between the rate of strain and the shearing stress. The non-Newtonian fluids have a subclass called viscoelastic fluids, which combine viscous and elastic characteristics and are widely utilized in various industrial applications. The viscoelastic properties of fluids are characterized by three constants, namely, the coefficient of viscosity μ, the retardation time $\overline{\lambda}_1$, and the relaxation time $\overline{\lambda}_2$, and termed as Oldroydian fluid, when both stress relaxation time and strain retardation time characteristics are prevalent (see [2, 20, 22] and references therein).

Various authors have studied the effects of viscoelastic parameters ($\overline{\lambda}_1$ and $\overline{\lambda}_2$) in non-Newtonian fluids in the Rayleigh-Bénard geometry and have discussed both stationary and oscillatory modes of instability in linear stability regimes. However, because of the mathematical complexity involved in solving nonlinear stability problems for oscillatory cases, limited efforts have been put in by authors. Using nonlinear stability analysis, the problem of ferroconvection in a viscoelastic medium has been investigated for the stationary case by some authors. In [19], Laroze et al. demonstrated in their investigation of ferrofluid's chaotic convection that the system exhibits several changes in parameter space from an orderly to chaotic behavior. The stationary convection amplitude equation in a viscoelastic magnetic fluid was examined in [17]. This work was carried forward in [18] by examining the effect of rotation on the amplitude equation in stationary convection. Further, the authors of [4, 7, 16] studied the nonlinear oscillatory instability of thermal/thermohaline convection problems and discussed the effects of various parameters on the heat transfer rate (Nusselt number) utilizing the amplitude function from the complex Ginzburg-Landau equation. Siddheshwar et al. in [28] investigated the "effect of time-periodic vertical oscillations of the Rayleigh-Bénard system on nonlinear convection in viscoelastic liquids and they found that the effect of time-periodic vertical oscillations of the Rayleigh-Bénard system (also known as gravity modulation or g-jitter) is to increase the critical

Rayleigh number and to reduce the heat transport". Recently, Dhiman and Sood in [10] considered the nonlinear ferroconvection problem to discuss the effect of uniform vertical rotation and analyzed the effects of various factors on the Nusselt number for oscillatory instability. Further, for latest research on the nonlinear ferroconvection one may refer to [1, 30, 31] and references therein.

Nonlinear analysis is also helpful in understanding bifurcation, and hence, is one of the most active research fields these days. The mathematical study of variation in the shape of a particular family of curves is known as bifurcation theory. Bifurcations occur when equilibrium is reached by bigger invariant sets like periodic orbits. As elaborated by Karaaslan in [15], bifurcation is a spot or area, where something splits into two sections. In [22], Rosenblat has performed the bifurcation analysis utilizing nonlinear stability in a viscoelastic fluid layer. For more details and further progress on the subject matter of nonlinear convective instability in viscoelastic and ferrofluids, one may refer to [13, 26, 27] and references therein.

The investigations and analyses discussed above make it plentifully evident that oscillations may occur and arise at the onset of ferroconvection in a viscoelastic ferrofluid layer heated from below under the influence of the magnetic field. Hence, in view of the importance of oscillations in convective problems, the investigation of the nonlinear oscillatory instability of certain classes of non-Newtonian fluids is desired in understanding the phenomena. As far as the authors' knowledge, any work on nonlinear stability analysis of oscillatory modes in ferroconvection in the Oldroydian model has not yet been reported. In view of the above discussion, we are motivated to mathematically investigate the linear as well as nonlinear stability of the ferroconvective viscoelastic fluid layer. The first-order problem (the linear stability case), the second-order, which gives the expression for the Nusselt number, and the third-order problem, used to compute the amplitude from the derived Ginzburg-Landau equation, are discussed in the present analysis. In both linear as well as nonlinear stability analyses, the effects of various factors on the onset of convection are numerically determined and the outcomes are depicted graphically using Mathematica® software. The pitchfork bifurcation analysis is also presented at the end.

2. Mathematical formulation

Let us assume a horizontal infinite layer of incompressible viscoelastic ferromagnetic fluid held between two isothermal and dynamically free horizontal boundaries ($z = 0$ and $z = d$). These boundaries are kept at uniform temperatures T_1 and T_0 so that an adverse temperature gradient across the layer is maintained (as shown in Fig. 1) and a constant vertical magnetic field $\vec{H}$ permeates the system. We shall investigate the two-dimensional convective rolls; hence, the fluid flow parameters are taken to be independent of the y-coordinate. Here, our aim is to examine the stability of this physical configuration.

The fundamental equations governing the hydrodynamics of the above physical configuration under the Boussinesq approximation are the equations of continuity, equations of momentum, equation of energy, and equation of state, which are given as (see [10, 12, 25])

$$\nabla \cdot \vec{v} = 0, \tag{1}$$

$$\rho_0 \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right) \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right) \left[-\nabla p + \rho \vec{g} + \nabla \cdot \left(\vec{H}\vec{B}\right)\right] + \left(1 + \overline{\lambda}_2 \frac{\partial}{\partial t}\right) \left(\mu \nabla^2 \vec{v}\right), \tag{2}$$

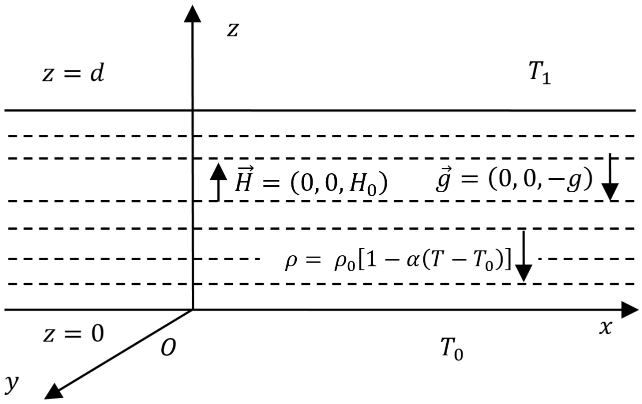

Fig. 1. Schematic sketch of the physical problem

$$\left[\rho_0 C_{V,H} - \eta_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \frac{DT}{Dt} + \eta_0 T \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \frac{D\vec{H}}{Dt} = K_1 \nabla^2 T, \tag{3}$$

$$\rho = \rho_0 \left[1 - \alpha \left(T - T_0\right)\right]. \tag{4}$$

In the above equations, $\vec{v} = (u, v, w)$ is the velocity, T is the temperature, $\vec{H}$ is the magnetic field, $\vec{g}$ is the external gravitational force, $\vec{B}$ is the magnetic induction, $\vec{M}$ is the magnetization and p is the pressure. Also, ρ, μ, $\overline{\lambda}_2$, $\overline{\lambda}_1$, α, K_1, η_0, and $C_{V,H}$ are, respectively, the fluid density, viscosity, strain retardation time, stress relaxation time, coefficient of volume expansion, thermal conductivity, magnetic permeability, and heat capacity at fixed volume and magnetic field. T_0 and ρ_0 are the values of temperature and density at the lower boundary and $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)$ is the material derivative.

In the absence of displacement current for a ferromagnetic fluid (non-conducting), the Maxwell's equations are given as

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{H} = 0, \qquad \vec{H} = \nabla\phi, \tag{5}$$

where $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic field, and ϕ is the magnetic scalar potential. Further, these quantities are related to the magnetic induction $\vec{B}$ as

$$\vec{B} = \eta_0 \left(\vec{H} + \vec{M}\right). \tag{6}$$

Equations $(5)_1$ and (6) are combined to yield

$$\nabla \cdot \left(\vec{H} + \vec{M}\right) = 0. \tag{7}$$

Assume that the magnetization and magnetic field are aligned, but depend upon the temperature and the magnitude of the magnetic field as

$$\vec{M} = \frac{\vec{H}}{H} M(H, T). \tag{8}$$

The magnetic equation of state in linearized form is given by

$$M = M_0 - K_2 \left(T - T_0\right) + \chi \left(H - H_0\right), \tag{9}$$

where M_0 is the magnetization at temperature T_0 and magnetic field H_0, $\chi = \left(\frac{\partial \vec{M}}{\partial \vec{H}}\right)_{H_0,T_0}$ is the magnetic susceptibility, and $K_2 = -\left(\frac{\partial \vec{M}}{\partial T}\right)_{H_0,T_0}$ is the pyromagnetic coefficient.

Initially, the basic state in the absence of motions, is given as

$$\begin{gathered}\vec{v} = (u, v, w) = \vec{v}_b = 0, \qquad T = T_b(z) = T_0 - \beta z, \qquad \rho = \rho_b(z), \qquad p = p_b(z), \\ \vec{H}_b = \left(H_0 - \frac{K_2 \beta z}{1+\chi}\right)\hat{k}, \qquad \vec{M}_b = \left(M_0 + \frac{K_2 \beta z}{1+\chi}\right)\hat{k}, \qquad H_0 + M_0 = {H_0}^{ext},\end{gathered} \tag{10}$$

where the subscript 'b' indicates the values of variables in their initial state and $\hat{k}$ is the unit vector in the vertical direction z. Further, the maintained uniform opposing temperature gradient is represented by $\beta = \frac{\Delta T}{d} = \frac{T_0 - T_1}{d}$.

To examine the instability of the above system at equilibrium, small perturbations are introduced to the basic state variables and the perturbed quantities are now represented as

$$\begin{gathered}\overline{\vec{v}} = \vec{v}_b + \vec{v}\,', \qquad \overline{T} = T_b(z) + \theta', \qquad \overline{p} = p_b(z) + \delta p', \\ \overline{\vec{H}} = \vec{H}_b(z) + \vec{H}', \qquad \overline{\vec{M}} = \vec{M}_b(z) + \vec{M}', \qquad \overline{\rho} = \rho_b(z) + \rho',\end{gathered} \tag{11}$$

where $\vec{v}\,' = (u', v', w')$, $\delta p'$, θ', $\vec{H}'$, ρ', and $\vec{M}'$ are perturbations in the initial velocity, pressure, temperature, magnetization, density, and intensity of the magnetic field, respectively.

To examine the *two-dimensional convective rolls instability*, utilizing the perturbed quantities (11) in (1)–(4) and (7)–(9) and dropping the primes for convenience in writing from the resulting equations, the following system of equations governing the perturbations is obtained:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{12}$$

$$\begin{aligned}\rho_0 \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) &= \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right)\left\{-\frac{\partial \delta p}{\partial x} + \eta_0 (H_0 + M_0)\frac{\partial H_x(z)}{\partial z}\right. \\ &\left. + \eta_0 \left[(H_x + M_x)\frac{\partial H_x(z)}{\partial x} + (H_z + M_z)\frac{\partial H_x(z)}{\partial z}\right]\right\} + \mu\left(1 + \overline{\lambda}_2 \frac{\partial}{\partial t}\right)\nabla^2 u,\end{aligned} \tag{13}$$

$$\begin{aligned}\rho_0 \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) &= \left(1 + \overline{\lambda}_1 \frac{\partial}{\partial t}\right)\left\{-\frac{\partial \delta p}{\partial z} + \eta_0 (H_0 + M_0)\frac{\partial H_z(z)}{\partial z}\right. \\ &+ \eta_0 \left[(H_x + M_x)\frac{\partial H_z(z)}{\partial x} + (H_z + M_z)\frac{\partial H_z(z)}{\partial z}\right] \\ &\left. + g\alpha\theta\rho_0 + \eta_0 (M_z + H_z)\frac{\partial}{\partial z}\left(H_0 - \frac{K_2 \beta z}{1+\chi}\right)\right\} + \mu\left(1 + \overline{\lambda}_2 \frac{\partial}{\partial t}\right)\nabla^2 w,\end{aligned} \tag{14}$$

$$\rho_0 C_1 \frac{\partial \theta}{\partial t} = K_1 \nabla^2 \theta + \rho_0 C_1 \beta w - \rho_0 C_1 \left(u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z}\right) + \Sigma(.,.), \tag{15}$$

$$\frac{\partial}{\partial x}(H_x + M_x) + \frac{\partial}{\partial z}(H_z + M_z) = 0, \tag{16}$$

$$H_x + M_x = \left(1 + \frac{M_0}{H_0}\right) H_x, \qquad \text{and} \qquad H_z + M_z = (1+\chi) H_z - K_2 \theta, \tag{17}$$

where $\rho_0 C_1 = \rho_0 C_{V,H} + K_2 H_0 \eta_0$. Also, the last term in (15) denoted by $\Sigma(.,.)$ consists of linear as well as nonlinear terms. Further, the approximation $K_2 \beta d \ll (1+\chi) H_0$ is used in deriving (17).

Now, eliminating δp between (13) and (14) and using the relations $(5)_3$ and (17), we get

$$\rho_0\left(1+\overline{\lambda}_1\frac{\partial}{\partial t}\right)\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)-\frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)\right]=$$

$$=\mu\left(1+\overline{\lambda}_2\frac{\partial}{\partial t}\right)\nabla^2\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)-\eta_0 K_2\left(1+\overline{\lambda}_1\frac{\partial}{\partial t}\right)\left[\frac{\partial\theta}{\partial x}\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial\theta}{\partial z}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)\right]$$

$$+\left(1+\overline{\lambda}_1\frac{\partial}{\partial t}\right)\left[\eta_0 K_2\beta\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\eta_0 {K_2}^2\beta}{(1+\chi)}\left(\frac{\partial\theta}{\partial x}\right)-\rho_0 g\alpha\left(\frac{\partial\theta}{\partial x}\right)\right]. \tag{18}$$

Using the stream function ψ and the perturbed magnetic potential ϕ defined by $u=\frac{\partial\psi}{\partial z}$, $w=-\frac{\partial\psi}{\partial x}$ and $H_x=\frac{\partial\phi}{\partial x}$, $H_z=\frac{\partial\phi}{\partial z}$ in (18), (15) and (16) and utilizing the following non-dimensional quantities (with hats) in the resulting equations:

$$\begin{aligned}&(x,y,z)=d\left(\hat{x},\hat{y},\hat{z}\right), && \psi=\kappa_T\hat{\psi}, && \overline{\lambda}_1=\hat{\lambda}_1\frac{d^2}{\kappa_T}, && \theta=\Delta T\cdot\hat{\theta}=\beta d\hat{\theta},\\ &t=\hat{t}\,\frac{d^2}{\kappa_T}, && v=\frac{\kappa_T\hat{v}}{d}, && \overline{\lambda}_2=\hat{\lambda}_2\,\frac{d^2}{\kappa_T}, && \phi=\hat{\phi}\,\frac{K_2\beta d^2}{(1+\chi)},\end{aligned} \tag{19}$$

the following system of non-dimensional equations is obtained (after dropping the hats):

$$\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}-\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2\right]\nabla^2\psi+\left(1+\lambda_1\frac{\partial}{\partial t}\right)R\left(1+M_1\right)\frac{\partial\theta}{\partial x}$$

$$-RM_1\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)=\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{1}{\sigma}J\left(\psi,\nabla^2\psi\right)-RM_1J\left(\theta,\frac{\partial\phi}{\partial z}\right)\right], \tag{20}$$

$$\frac{\partial\psi}{\partial x}+\left(\frac{\partial}{\partial t}-\nabla^2\right)\theta=J\left(\psi,\theta\right), \tag{21}$$

$$\frac{\partial\theta}{\partial z}-\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0, \tag{22}$$

where $\sigma=\frac{\nu}{\kappa_T}$, $M_1=\frac{\eta_0{K_2}^2\beta}{\rho_0 g\alpha(1+\chi)}$, and $R=\frac{g\alpha\beta d^4}{\nu\kappa_T}$ are the Prandtl number, the magnetic number, and the Rayleigh number, respectively. Also in the above equations, $\nu=\frac{\mu}{\rho_0}$ denotes the kinematic viscosity and $M_3=\left(1+\frac{M_0}{H_0}\right)\frac{1}{(1+\chi)}$ is the nonlinearity parameter of magnetization. Further, λ_2 and λ_1 are the non-dimensional strain retardation and stress relaxation times, respectively.

The nonlinear terms in the aforementioned equations are represented by $J(.,.)$ in the form of Jacobians. Further, all the terms in the expression $\Sigma\left(.,.\right)$ in (15) after non-dimensionalization vanish, because all these terms have a multiplicative factor $M_2\left(=\frac{\eta_0 T_0{K_2}^2}{\rho_0 C_1(1+\chi)}\right)$, which is taken to be very small of the order of 10^{-5}, see [10, 12]. The system of equations (20)–(22) is solved with respect to the following boundary conditions:

$$\psi=\frac{\partial^2\psi}{\partial z^2}=\theta=\frac{\partial\phi}{\partial z}=0\quad\text{at}\quad z=0\text{ and }1\quad(\text{stress free and isothermal boundaries}). \tag{23}$$

3. Mathematical analysis

Following the analysis of Dhiman and Sood in [10], the variables appearing in (20)–(22) are expressed in power of small perturbations ε as follows:

$$\begin{aligned}&R=R_c+\varepsilon^2R_2+\ldots, && \psi=\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3+\ldots,\\ &\theta=\varepsilon\theta_1+\varepsilon^2\theta_2+\varepsilon^3\theta_3+\ldots, && \phi=\varepsilon\phi_1+\varepsilon^2\phi_2+\varepsilon^3\phi_3+\ldots,\end{aligned} \tag{24}$$

where R_c is the critical Rayleigh number. Considering the slow time scale s and the fast time scale τ, in order to have the anticipated frequency shift along the bifurcation solution as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \varepsilon^2 \frac{\partial}{\partial s}. \tag{25}$$

Now, substituting the expansions (24) and (25) into the non-dimensional equations (20)–(22), we obtain the following equations:

$$\begin{aligned}
&\left\{\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_1 + (1+M_1)\,R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial\theta_1}{\partial x}\right.\\
&\left.-R_cM_1\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right)\right\}\varepsilon + \left\{\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_2\right.\\
&+\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)\,R_c\frac{\partial\theta_2}{\partial x} - R_cM_1\frac{\partial}{\partial x}\left(\frac{\partial\phi_2}{\partial z}\right)\right] - \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[\frac{1}{\sigma}J\left(\psi_1,\nabla^2\psi_1\right)\right.\\
&\left.\left.-R_cM_1J\left(\theta_1,\frac{\partial\phi_1}{\partial z}\right)\right]\right\}\varepsilon^2 + \left\langle\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_3\right.\\
&+\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)\,R_c\frac{\partial\theta_3}{\partial x} - R_cM_1\frac{\partial}{\partial x}\left(\frac{\partial\phi_3}{\partial z}\right)\right] - \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left\{\frac{1}{\sigma}\left[J\left(\psi_1,\nabla^2\psi_2\right)\right.\right.\\
&\left.\left.+J\left(\psi_2,\nabla^2\psi_1\right)\right] - R_cM_1\left[J\left(\theta_1,\frac{\partial\phi_2}{\partial z}\right) + J\left(\theta_2,\frac{\partial\phi_1}{\partial z}\right)\right]\right\}\\
&+\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial s} + \frac{1}{\sigma}\lambda_1\frac{\partial}{\partial\tau}\frac{\partial}{\partial s} - \lambda_2\frac{\partial}{\partial s}\nabla^2\right]\nabla^2\psi_1 + (1+M_1)\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2\right.\\
&\left.\left.+\lambda_1R_c\frac{\partial}{\partial s}\right]\frac{\partial\theta_1}{\partial x} - M_1\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2 + \lambda_1R_c\frac{\partial}{\partial s}\right]\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right)\right\rangle\varepsilon^3 = 0,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\left[\frac{\partial\psi_1}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_1\right]\varepsilon + \left[\frac{\partial\psi_2}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_2 - J\left(\psi_1,\theta_1\right)\right]\varepsilon^2\\
&+\left\{\frac{\partial\psi_3}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_3 + \left[\frac{\partial\theta_1}{\partial s} - J\left(\psi_1,\theta_2\right) - J\left(\psi_2,\theta_1\right)\right]\right\}\varepsilon^3 = 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\left[\frac{\partial\theta_1}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_1\right]\varepsilon + \left[\frac{\partial\theta_2}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_2\right]\varepsilon^2\\
&+\left[\frac{\partial\theta_3}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_3\right]\varepsilon^3 = 0.
\end{aligned} \tag{28}$$

In order to have first, second and third order stability problems, we can compare the coefficients of ε, ε^2 and ε^3 from (26)–(28). In the following analysis, each of these stability problems is discussed separately.

The problem of *first-order* stability is given by the following system of equations:

$$\begin{aligned}
&\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_1 + (1+M_1)\,R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial\theta_1}{\partial x}\\
&-R_cM_1\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right) = 0,
\end{aligned} \tag{29}$$

$$\frac{\partial\psi_1}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_1 = 0, \tag{30}$$

$$\frac{\partial\theta_1}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_1 = 0. \tag{31}$$

Following [10] and [16], the solution of (29)–(31) in view of (23) is represented in the following periodic forms as:

$$\begin{aligned}\theta_1 &= \left(A_1(s)e^{i\omega\tau} + \overline{A}_1(s)e^{-i\omega\tau}\right)\cos ax \sin \pi z,\\ \psi_1 &= \left(B_1(s)e^{i\omega\tau} + \overline{B}_1(s)e^{-i\omega\tau}\right)\sin ax \sin \pi z,\\ \phi_1 &= \left(C_1(s)e^{i\omega\tau} + \overline{C}_1(s)e^{-i\omega\tau}\right)\cos ax \cos \pi z.\end{aligned} \tag{32}$$

Here, a and ω stand for the wavenumber and frequency, respectively, corresponding to the Rayleigh number R, whereas the overbar $(^{-})$ denotes a complex conjugate.

The relationships between the undetermined amplitudes A_1, B_1, and C_1 appearing in (32) are given by

$$A_1(s) = -\frac{a}{c+i\omega}B_1(s), \qquad C_1(s) = \frac{\pi a}{(c+i\omega)\left(M_3a^2+\pi^2\right)}B_1(s), \tag{33}$$

where

$$c = a^2 + \pi^2. \tag{34}$$

Substituting solutions (32) in (29)–(31) and eliminating the undetermined amplitudes $A_1(s)$, $B_1(s)$, and $C_1(s)$ with the help of (33), we get a complex equation. The real and imaginary parts of this equation yield the expressions for the thermal Rayleigh number and its corresponding periodic oscillation frequency given as

$$R^{\text{Ov}} = \left\{\frac{c^3}{a^2}\left(\frac{1}{1+\omega^2{\lambda_1}^2}\right) - \frac{\lambda_1^2 c\omega^4 - \left[\sigma\lambda_1\lambda_2c^3 + \sigma\left(\lambda_1-\lambda_2\right)c^2 - c\right]\omega^2}{a^2\sigma\left(1+\omega^2{\lambda_1}^2\right)}\right\}\frac{M_3a^2+\pi^2}{M_3\left(1+M_1\right)a^2+\pi^2} \tag{35}$$

and

$$\omega = \sqrt{\frac{c\sigma\left(\lambda_1-\lambda_2\right)-(1+\sigma)}{\lambda_1(\lambda_1+\lambda_2\sigma)}}. \tag{36}$$

Using (36) in (35), we get the following expression for the oscillatory Rayleigh number:

$$R^{\text{Ov}} = \frac{M_3a^2+\pi^2}{M_3\left(1+M_1\right)a^2+\pi^2}\cdot\frac{\lambda_1+\lambda_2\sigma}{\lambda_1+\lambda_2\sigma+\lambda_1\left[c\sigma\left(\lambda_1-\lambda_2\right)-(\sigma+1)\right]}\cdot\left\{\frac{c^3}{a^2}+\frac{\left[c^2\sigma\left(\lambda_1-\lambda_2\right)-(\sigma+1)\,c\right]\left[c^2\lambda_1\lambda_2\sigma\left(\lambda_1+\lambda_2\sigma\right)+c\sigma^2\lambda_2\left(\lambda_1-\lambda_2\right)+\left(\lambda_1+1\right)-\sigma\left(\lambda_1-\lambda_2\right)\right]}{a^2\sigma\lambda_1\left(1+\sigma\lambda_2\right)^2}\right\}. \tag{37}$$

From (36), one can conclude that for $\lambda_1 \le \lambda_2$, the oscillatory motions are not admissible as the frequency ω cannot be a complex quantity. This means that for oscillatory convection to appear, the following inequality must hold good:

$$\lambda_1 > \lambda_2, \tag{38}$$

which is the same condition obtained by Basu and Layek in [2] for classical Newtonian fluids. Also for the viscoelastic fluid layer in the absence of ferromagnetic effects (i.e., when $M_1 = M_3 = 0$), formula (37) yields the following form of the oscillatory thermal Rayleigh number:

$$R^{\text{Ov}} = \frac{\lambda_1+\lambda_2\sigma}{\left(\lambda_1+\lambda_2\sigma\right)+\lambda_1\left[c\sigma\left(\lambda_1-\lambda_2\right)-(\sigma+1)\right]}\cdot\left\{\frac{c^3}{a^2}+\frac{\left[c^2\sigma\left(\lambda_1-\lambda_2\right)-(\sigma+1)\,c\right]\left[c^2\lambda_1\lambda_2\sigma\left(\lambda_1+\lambda_2\sigma\right)+c\sigma^2\lambda_2\left(\lambda_1-\lambda_2\right)+\left(\lambda_1+1\right)-\sigma\left(\lambda_1-\lambda_2\right)\right]}{a^2\sigma\lambda_1\left(1+\sigma\lambda_2\right)^2}\right\}. \tag{39}$$

Similar expression was obtained in [2].

Further, expression (35) for the steady convection (when $\omega = 0$) reduces to the following thermal formula for the Rayleigh number:

$$R^{\text{Steady}} = \frac{c^3(M_3 a^2 + \pi^2)}{a^2\left[M_3\left(1+M_1\right)a^2 + \pi^2\right]}, \tag{40}$$

the same result obtained by Finlayson in [12] for the case of a ferrofluid layer. Also when $M_1 = M_3 = 0$ (for an ordinary viscous fluid layer), the same value of the Rayleigh number was attained in [6] by Chandrasekhar for stationary convection as

$$R^{\text{Steady}} = \frac{c^3}{a^2}. \tag{41}$$

For the *second-order* problem, equations (26)–(28) yield the following system of equations:

$$\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_2 + \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[\left(1+M_1\right)R_c\frac{\partial\theta_2}{\partial x}\right.$$
$$\left. - R_c M_1 \frac{\partial}{\partial x}\left(\frac{\partial\phi_2}{\partial z}\right)\right] = \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[\frac{1}{\sigma}J\left(\psi_1,\nabla^2\psi_1\right) - R_c M_1 J\left(\theta_1,\frac{\partial\phi_1}{\partial z}\right)\right], \tag{42}$$

$$\frac{\partial\psi_2}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_2 = J\left(\psi_1,\theta_1\right), \tag{43}$$

$$\frac{\partial\theta_2}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_2 = 0, \tag{44}$$

where the Jacobians representing the nonlinear terms $J\left(\psi_1,\nabla^2\psi_1\right)$, $J\left(\theta_1,\frac{\partial\phi_1}{\partial z}\right)$ and $J\left(\psi_1,\theta_1\right)$ yield the following values on expansion:

$$J\left(\psi_1,\nabla^2\psi_1\right) = 0, \qquad J\left(\theta_1,\frac{\partial\phi_1}{\partial z}\right) = 0,$$
$$J\left(\psi_1,\theta_1\right) = \frac{\pi a}{2}\left[\begin{array}{c}\overline{A}_1(s)B_1(s) + A_1(s)\overline{B}_1(s) + \\ A_1(s)B_1(s)e^{2i\omega\tau} + \overline{A}_1(s)\overline{B}_1(s)e^{-2i\omega\tau}\end{array}\right]\sin 2\pi z. \tag{45}$$

Now, for this problem, the temperature is expressed as

$$\theta_2 = \left(\theta_{20} + \theta_{22}e^{2i\omega\tau} + \overline{\theta}_{22}e^{-2i\omega\tau}\right)\sin 2\pi z. \tag{46}$$

Using (45) and (46), the expression for the magnetic potential can be written as

$$\phi_2 = \left(\phi_{20} + \phi_{22}e^{2i\omega\tau} + \overline{\phi}_{22}e^{-2i\omega\tau}\right)\cos 2\pi z, \tag{47}$$

where θ_{20} and ϕ_{20} are the temperature and magnetic scalar potential fields independent of τ, and θ_{22} and ϕ_{22} are the temperature and magnetic scalar potential fields associated with the frequency 2ω. Utilizing the expansions given by (45)–(47) in (42)–(44), the second-order stability problem now takes the following form:

$$\psi_2 = 0, \quad \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_2 = J\left(\psi_1,\theta_1\right), \quad \frac{\partial\theta_2}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_2 = 0. \tag{48}$$

Using (45)–(47) in (48), substituting the values of $A_1(s)$, $B_1(s)$ and $C_1(s)$ from (33) and then solving the resulting equations, the following solutions in the form of amplitude are obtained:

$$\psi_{20} = 0, \quad \theta_{20} = -\frac{a^2 c}{4\pi\left(c^2+\omega^2\right)}\left|B_1(s)\right|^2, \quad \phi_{20} = \frac{a^2 c}{8\pi^2\left(c^2+\omega^2\right)}\left|B_1(s)\right|^2, \tag{49}$$

$$\psi_{22} = 0, \quad \theta_{22} = \frac{-a^2\pi}{\left(8\pi^2+4i\omega\right)\left(c+i\omega\right)}B_1(s)^2, \quad \phi_{22} = \frac{a^2}{\left(16\pi^2+8i\omega\right)\left(c+i\omega\right)}B_1(s)^2. \tag{50}$$

We know that the expression for the averaged Nusselt number Nu is given by

$$\text{Nu} = 1 - \varepsilon^2\left(\frac{\partial\theta_2}{\partial z}\right)_{z=0}, \tag{51}$$

the value of which can be obtained using (46) and (49)–(50), which obviously involves the undetermined amplitude $B_1(s)$. This amplitude will be obtained by solving the Ginzburg-Landau equation in the following analysis.

As discussed earlier, equations (26)–(28) yield the *third-order* problem governed by the following set of equations:

$$\begin{aligned}
&\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_3 + \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[\left(1+M_1\right)R_c\frac{\partial\theta_3}{\partial x} - \right.\\
&\left. R_c M_1\frac{\partial}{\partial x}\left(\frac{\partial\phi_3}{\partial z}\right)\right] = \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left\{\frac{1}{\sigma}\left[J\left(\psi_1,\nabla^2\psi_2\right)+J\left(\psi_2,\nabla^2\psi_1\right)\right]\right.\\
&\left. -R_c M_1\left[J\left(\theta_1,\frac{\partial\phi_2}{\partial z}\right)+J\left(\theta_2,\frac{\partial\phi_1}{\partial z}\right)\right]\right\} - \left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial s}\right.\\
&\left. +\frac{1}{\sigma}\lambda_1\frac{\partial}{\partial\tau}\frac{\partial}{\partial s} - \lambda_2\frac{\partial}{\partial s}\nabla^2\right]\nabla^2\psi_1 - \left(1+M_1\right)\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2 + \lambda_1 R_c\frac{\partial}{\partial s}\right]\frac{\partial\theta_1}{\partial x}\\
&+M_1\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2 + \lambda_1 R_c\frac{\partial}{\partial s}\right]\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right),
\end{aligned} \tag{52}$$

$$\frac{\partial\psi_3}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_3 = J\left(\psi_1,\theta_2\right) + J\left(\psi_2,\theta_1\right) - \frac{\partial\theta_1}{\partial s}, \tag{53}$$

$$\frac{\partial\theta_3}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_3 = 0. \tag{54}$$

Using $\psi_2 = 0$ from $(48)_1$, the non-linear Jacobian terms of (52) and (53) are obtained as

$$J\left(\psi_1,\nabla^2\psi_2\right) + J\left(\psi_2,\nabla^2\psi_1\right) = 0, \tag{55}$$

$$\begin{aligned}
J\left(\theta_1,\frac{\partial\phi_2}{\partial z}\right) + J\left(\theta_2,\frac{\partial\phi_1}{\partial z}\right) &= \frac{a^4\pi^2}{c^2+\omega^2}\left[\frac{c}{4\pi^2(c+i\omega)} + \frac{1}{8\pi^2+4i\omega} - \frac{c}{4\left(c+i\omega\right)\left(M_3a^2+\pi^2\right)}\right.\\
&\left. -\frac{\pi^2}{(M_3a^2+\pi^2)(8\pi^2+4i\omega)}\right]B_1(s)\left|B_1(s)\right|^2 e^{i\omega\tau}\sin ax\sin\pi z,
\end{aligned} \tag{56}$$

$$\begin{aligned}
J\left(\psi_1,\theta_2\right) + J\left(\psi_2,\theta_1\right) &= \pi^2 a^3\left[\frac{c}{4\pi^2(c^2+\omega^2)} + \frac{1}{\left(c+i\omega\right)\left(8\pi^2+4i\omega\right)}\right]\\
&B_1(s)\left|B_1(s)\right|^2 e^{i\omega\tau}\cos ax\sin\pi z.
\end{aligned} \tag{57}$$

In view of the values of the sum of the Jacobian terms appearing in (55)–(57), let us consider the solutions of the third-order problem (52)–(54) in the following forms:

$$\theta_3 = A_3(s)e^{i\omega\tau}\cos ax \sin \pi z, \tag{58}$$

$$\psi_3 = B_3(s)e^{i\omega\tau}\sin ax \sin \pi z, \tag{59}$$

$$\phi_3 = C_3(s)e^{i\omega\tau}\cos ax \cos \pi z. \tag{60}$$

Upon using the values of Jacobians from (55) and $J(\psi_2, \theta_1) = 0$ (in view of $\psi_2 = 0$) in (52)–(53), equations (52)–(54) assume the following forms:

$$\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_3 + \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)R_c\frac{\partial\theta_3}{\partial x} - R_cM_1\frac{\partial}{\partial x}\left(\frac{\partial\phi_3}{\partial z}\right)\right] = R_{31}, \tag{61}$$

$$\frac{\partial\psi_3}{\partial x} + \left(\frac{\partial}{\partial\tau} - \nabla^2\right)\theta_3 = R_{32}, \tag{62}$$

$$\frac{\partial\theta_3}{\partial z} - \left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi_3 = 0, \tag{63}$$

where

$$R_{31} = -R_cM_1\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[J\left(\theta_1, \frac{\partial\phi_2}{\partial z}\right) + J\left(\theta_2, \frac{\partial\phi_1}{\partial z}\right)\right] - \left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial s} + \frac{1}{\sigma}\lambda_1\frac{\partial}{\partial\tau}\frac{\partial}{\partial s} - \lambda_2\frac{\partial}{\partial s}\nabla^2\right]\nabla^2\psi_1 - (1+M_1)\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2 + \lambda_1R_c\frac{\partial}{\partial s}\right]\frac{\partial\theta_1}{\partial x} + M_1\left[\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_2 + \lambda_1R_c\frac{\partial}{\partial s}\right]\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right) \tag{64}$$

and

$$R_{32} = J(\psi_1, \theta_2) - \frac{\partial\theta_1}{\partial s}. \tag{65}$$

The above system of equations (61)–(63) can be written in a matrix form ($AX = B$) as

$$\begin{bmatrix} A_{11} & A_{12} & -\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)R_cM_1\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau} - \nabla^2\right) & 0 \\ 0 & \frac{\partial}{\partial z} & -\left(M_3\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \end{bmatrix} \begin{bmatrix} \psi_3 \\ \theta_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \\ 0 \end{bmatrix}, \tag{66}$$

where

$$A_{11} = \left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau} - \left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2,\ A_{12} = \left(1+\lambda_1\frac{\partial}{\partial\tau}\right)(1+M_1)R_c\frac{\partial}{\partial x}.$$

The matrix operator A is said to be a self-adjoint operator if and only if it is symmetric. However, from (66), it is clear that the matrix operator A is not a self-adjoint operator; hence, for a non-self-adjoint operator, the solvability condition cannot be applied. Therefore, to overcome this difficulty, let us construct a self-adjoint operator by reducing the order of the matrix, as follows.

Applying the operator $\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)$ to both sides of (61) and then using the relation derived from (63) as $\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi_3=\frac{\partial\theta_3}{\partial z}$, we get the following equation:

$$\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau}-\left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2\psi_3+R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)$$
$$\left[(1+M_1)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-M_1\frac{\partial^2}{\partial z^2}\right]\frac{\partial\theta_3}{\partial x}=\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)R_{31}. \tag{67}$$

Next, let us write (67) and (62) in the matrix form

$$\begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \frac{\partial}{\partial x} & \left(\frac{\partial}{\partial\tau}-\nabla^2\right)\end{bmatrix}\begin{bmatrix}\psi_3\\ \theta_3\end{bmatrix}=\begin{bmatrix}\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)R_{31}\\ R_{32}\end{bmatrix}, \tag{68}$$

where

$$\hat{A}_{11}=\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau}-\left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2,$$
$$\hat{A}_{12}=R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-M_1\frac{\partial^2}{\partial z^2}\right]\frac{\partial}{\partial x}.$$

Applying the elementary row operation

$$\mathbf{R}_2\to\mathbf{R}_2\times\left\{R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-M_1\frac{\partial^2}{\partial z^2}\right]\right\},$$

the system (68) takes the following form:

$$\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}\psi_3\\ \theta_3\end{bmatrix}=\begin{bmatrix}\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)R_{31}\\ R_c\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left[(1+M_1)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-M_1\frac{\partial^2}{\partial z^2}\right]R_{32}\end{bmatrix}, \tag{69}$$

where

$$\tilde{A}_{11}=\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{\sigma}\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial\tau}-\left(1+\lambda_2\frac{\partial}{\partial\tau}\right)\nabla^2\right]\nabla^2,$$
$$\tilde{A}_{12}=\tilde{A}_{21}=\left[R_c\left(1+M_1\right)\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-R_cM_1\frac{\partial^2}{\partial z^2}\right]\frac{\partial}{\partial x},$$
$$\tilde{A}_{22}=\left[R_c\left(1+M_1\right)\left(1+\lambda_1\frac{\partial}{\partial\tau}\right)\left(M_3\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)-M_1\frac{\partial^2}{\partial z^2}\right]\left(\frac{\partial}{\partial\tau}-\nabla^2\right).$$

From (69), it is clear that the matrix operator is symmetric, and hence, a self-adjoint matrix operator. Further, it should be mentioned here that the non-self-adjoint matrix operator in system (66) is reduced to a self-adjoint operator matrix by an elementary matrix operation. And since the reduced self-adjoint matrix operator is a second-order system that arises from the non-self-adjoint third-order system, we would expect the same Ginzburg-Landau equation as a result of using the solvability in the two systems. So, using the solutions (58)–(60) and the values of Jacobians from (55)–(57) in (69), following the analyses in [10] and [16], and finally invoking the solvability criteria, the desired Ginzburg-Landau equation is obtained as follows:

$$\gamma B_1'(s)-FB_1(s)+k\left|B_1(s)\right|^2B_1(s)=0, \tag{70}$$

where $'$ denotes the derivative of the quantity and γ, F and k are given by

$$\gamma = \left[1 - \frac{R_c a^2 \sigma \lambda_1}{c\,(c+i\omega)\,(1+2i\omega\lambda_1)} + \frac{R_c a^2 \sigma}{c\,(c+i\omega)^2} \frac{1+i\omega\lambda_1}{1+2i\omega\lambda_1} \frac{M_3 a^2\,(1+M_1)+\pi^2}{M_3 a^2+\pi^2} + \frac{\lambda_2 c \sigma}{1+2i\omega\lambda_1}\right], \tag{71}$$

$$F = \left[\frac{R_2 a^2 \sigma}{c\,(c+i\omega)} \frac{M_3 a^2\,(1+M_1)+\pi^2}{M_3 a^2+\pi^2} \frac{1+i\omega\lambda_1}{1+2i\omega\lambda_1}\right], \tag{72}$$

$$\begin{aligned} k =& \frac{R_c a^4 \pi^2 \sigma M_1}{c\,(c^2+\omega^2)} \frac{1+i\omega\lambda_1}{1+2i\omega\lambda_1} \left[\frac{c}{4\,(c+i\omega)\,(M_3 a^2+\pi^2)} + \frac{\pi^2}{(8\pi^2+4i\omega)\,(M_3 a^2+\pi^2)} \right. \\ & \left. - \frac{c}{4\pi^2\,(c+i\omega)} - \frac{1}{8\pi^2+4i\omega}\right] + \frac{R_c a^4 \pi^2 \sigma}{c+i\omega} \frac{M_3 a^2\,(1+M_1)+\pi^2}{M_3 a^2+\pi^2} \frac{1+i\omega\lambda_1}{1+2i\omega\lambda_1} \\ & \left[\frac{1}{4\pi^2\,(c^2+\omega^2)} + \frac{1}{(8\pi^2+4i\omega)\,c\,(c+i\omega)}\right]. \end{aligned} \tag{73}$$

One should mention here that the Ginzburg-Landau equations, derived by Bhadauria and Kiran in [4] for non-ferro viscoelastic fluids and by Dhiman and Sood in [10] for non-viscoelastic ferrofluids, can be easily deduced from (70)–(73) by neglecting the respective effects.

Now, $B_1(s)$ can be expressed in phase-amplitude form as

$$B_1(s) = |B_1(s)|\, e^{i\Phi}. \tag{74}$$

Using (74) in (70) and comparing the real and imaginary components of the resulting equations, a pair of equations involving the amplitude $|B_1(s)|$ is obtained as

$$\left(|B_1(s)|^2\right)' - 2\, p_r\, |B_1(s)|^2 + 2\, l_r\, |B_1(s)|^4 = 0, \tag{75}$$

$$(ph(B(s)))' = p_i - l_i\, |B_1(s)|^2\,, \tag{76}$$

where ${\gamma_1}^{-1}F = p_r + ip_i$ and ${\gamma_1}^{-1}k_1 = l_r + il_i$, and $ph\,(.)$ denotes the phase shift.

Following the work [11] in terms of amplitude $B_0(s)$, the growth of $|B_1(s)|$ with time can be represented as

$$|B_1(s)|^2 = {B_0}^2(s) \left\{\frac{l_r}{p_r}{B_0}^2(s) + \left[1 - \frac{l_r}{p_r}{B_0}^2(s)\right] e^{-2p_r s}\right\}^{-1}, \quad p_r > 0,\ l_r > 0. \tag{77}$$

It is clear from (77) that as $s \to -\infty$, $|B_1(s)| \to 0$, and as $s \to +\infty$, then $|B_1(s)|$ propagates in the direction of $\sqrt{p_r/l_r}$, whenever $0 < B_0 < \sqrt{p_r/l_r}$, and decreases towards $\sqrt{p_r/l_r}$, whenever $B_0 > \sqrt{p_r/l_r}$. It is obvious that when $p_r < 0$, the point $B_1(s) = 0$ is the only point of equilibrium that is stable. When $p_r = 0$, the only stable equilibrium position is at the origin. Further, when $p_r > 0,\ l_r > 0$, $|B_1(s)| = 0$ is still an equilibrium point. However, it deviates from equilibrium and two new symmetrically situated stable points at $|B_1(s)| = \pm\sqrt{p_r/l_r}$ appear on either side of $|B_1(s)| = 0$ (shown in Figs. 11 and 13). This kind of bifurcation is known as a supercritical pitchfork bifurcation.

Using (33), (34), (46), (49), and (50), the value of the horizontally averaged Nusselt number from (51) is given by

$$\mathrm{Nu} = 1 + \left[\frac{a^2 c}{2\,(c^2+\omega^2)} + \frac{2\pi^2 a^2}{\sqrt{64\pi^4 + 16\omega^2\sqrt{c^2+\omega^2}}}\right] |B_1(s)|^2\,. \tag{78}$$

Following the works [3,5], wherein the authors have commented that ”the Ginzburg-Landau equation is the Bernoulli equation and obtaining its analytic solution is difficult due to its non-autonomous nature”. So, this is solved numerically subjected to the suitable initial condition $B_0 = b_0$, where b_0 is the appropriately taken initial convection amplitude. In the present analysis, $R_2 = R_c$ (the initial condition, where R_c is the value of the threshold Rayleigh number, at which the convection just starts) is assumed to keep the parameters to the minimum.

4. Results and discussions

The effects of magnetic number M_1, nonlinearity of magnetization parameter M_3, strain retardation time λ_2, and stress relaxation time λ_1 on ferroconvection in a viscoelastic fluid layer is studied in this paper. From earlier studies (e.g., [8, 12]), it is clear that the oscillations are not permitted in the ferroconvection problem, and hence, only the stationary instability is allowed. However, in the present analysis, it is shown that the oscillatory convection can exist in the ferromagnetic viscoelastic fluid layer, when $\lambda_1 > \lambda_2$. The stress relaxation time λ_1 defines the time, which a system takes to relax in response to certain changes in the environment, whereas the strain retardation time λ_2 specifies the time required by the system to regain its equilibrium position after imposition of stress. Both parameters are crucial for defining the properties of a viscoelastic fluid.

The present section is divided into two subsections: linear stability analysis and weakly non-linear stability analysis. We shall discuss each of these separately.

4.1. Linear stability analysis

In linear stability analysis, the values of the thermal Rayleigh numbers for stationary R^{Steady} and oscillatory R^{Ov} cases were numerically calculated corresponding to the wave number a for the constant values of $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$, $M_1 = 10$, and $M_3 = 5$ (cf. [2, 12]), using the Mathematica® software. The outcomes are presented through graphs in Figs. 2–5.

Fig. 2 depicts the effect of M_3 on stationary ${R_c}^{\text{Steady}}$ and oscillatory ${R_c}^{\text{Ov}}$ Rayleigh numbers plotted as a function of the wave number a. From the results, we observed that M_3 has destabilizing influence on the onset of convection. ”This effect may be due to the large value of the pyromagnetic coefficient or due to the large temperature gradient” as claimed by Dhiman and Sood in [10]. Similarly, from Fig. 3, the effect of M_1 on stationary ${R_c}^{\text{Steady}}$ and oscillatory ${R_c}^{\text{Ov}}$ Rayleigh numbers plotted for varying a also indicates destabilizing influence of M_1 on the onset of convection. In [10], Dhiman and Sood also claimed that ”this behavior of the ferrofluid convective system may be due to the increase in the destabilizing magnetic force, as heat is being now transported more efficiently in magnetic ferrofluids as compared to ordinary fluids”. Further, from the variations presented in Figs. 2 and 3, it is observed that the oscillatory mode of ferroconvection is the preferred mode in the present problem.

In Fig. 4, the effect of λ_1 on $R^{\text{Ov}} = R^{\text{Ov}}(a)$ is depicted graphically in view of the condition $\lambda_1 > \lambda_2$, see (38). From the variations, it is observed that λ_1 has destabilizing influence on the onset of oscillatory instability. While Fig. 5 depicts the effect of λ_2 on R^{Ov} as a function of a, which validates the stabilizing effect of λ_2 as per the claim made by Basu and Layek in [2].

4.2. Weakly non-linear stability analysis

For the case of weakly nonlinear instability, the interpolating function describing amplitude $B_1(s)$ for the constant values of the parameters (cf. [21]): $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$, $M_1 = 10$, and $M_3 = 5$, is obtained from the complex Ginzburg-Landau equation. Consequently,

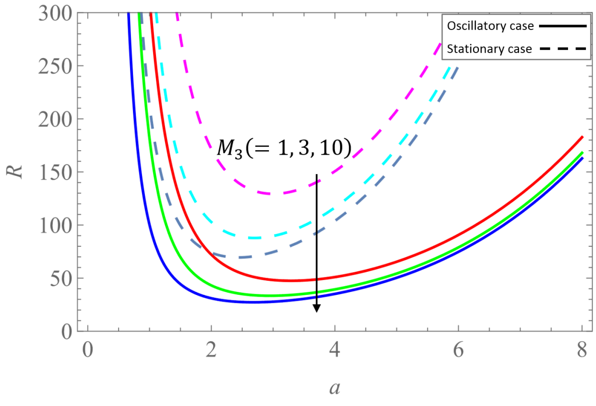

Fig. 2. R as a function of a for $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$ and $M_1 = 10$

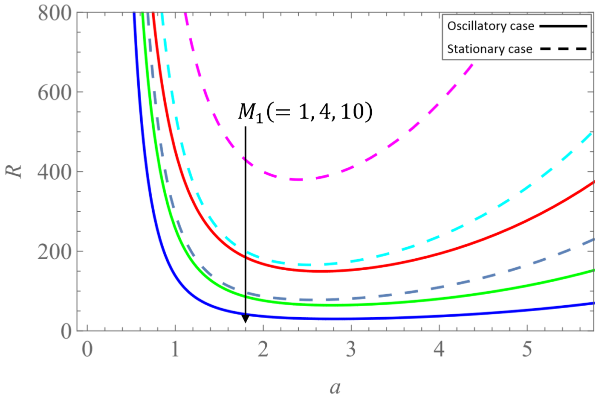

Fig. 3. R as a function of a for $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$ and $M_3 = 5$

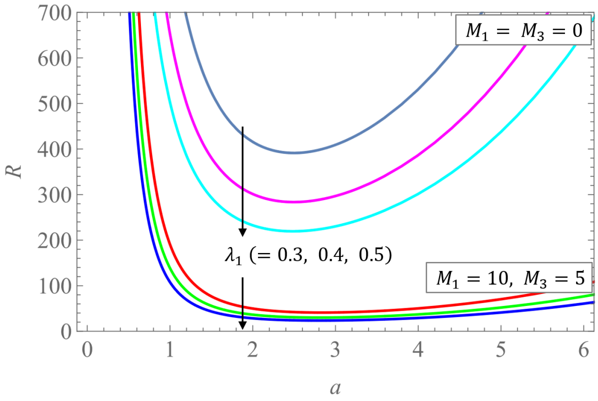

Fig. 4. R^{Ov} as a function of a for $\sigma = 10$ and $\lambda_2 = 0.1$

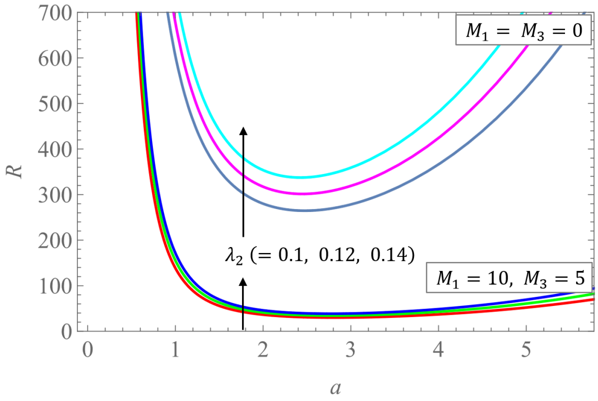

Fig. 5. R^{Ov} as a function of a for $\sigma = 10$ and $\lambda_1 = 0.4$

in Figs. 6–10, the variations of heat transfer rate Nu versus time s for a certain set of fixed values of parameters σ, λ_1, λ_2, M_1, and M_3, respectively, are depicted graphically.

In Fig. 6, the effect of σ is studied on the heat transfer rate Nu. It is observed that with increasing values of σ, the heat transfer rate decreases, cf. [21, 29].

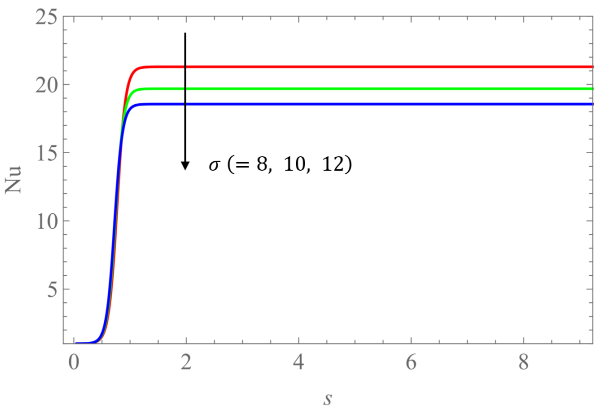

Fig. 6. Nu as a function of s for $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $M_1 = 10$ and $M_3 = 5$

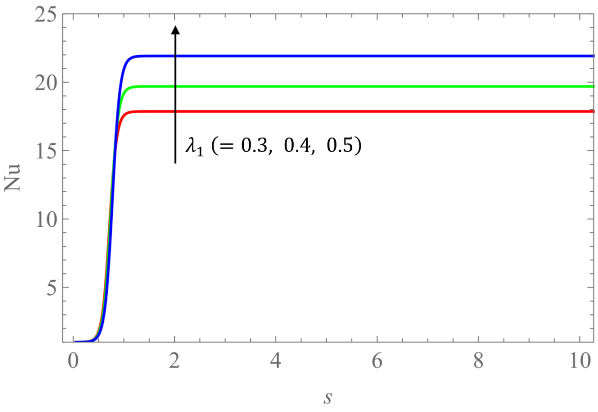

Fig. 7. Nu as a function of s for $\lambda_2 = 0.1$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

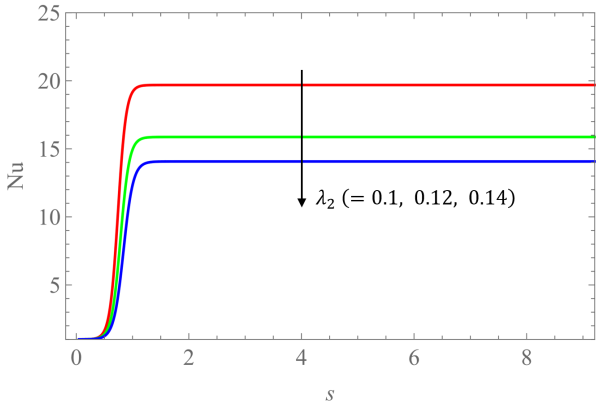

Fig. 8. Nu as a function of s for $\lambda_1 = 0.4$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

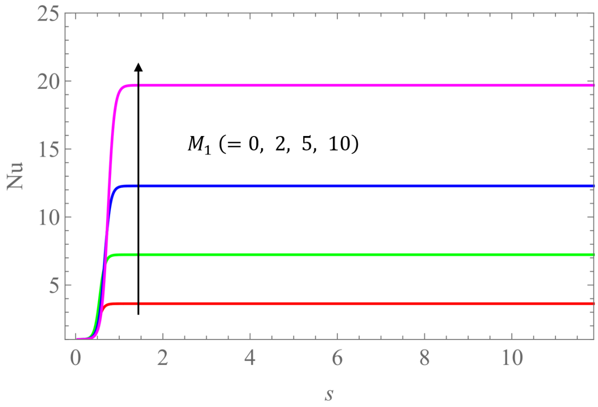

Fig. 9. Nu as a function of s for $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$ and $M_3 = 5$

The effects of λ_1 and λ_2 are studied on the heat transfer rate Nu in Figs. 7 and 8, respectively. From the variations shown in these graphs, it is found that λ_1 increases the heat transfer rate, whereas λ_2 decreases it. These findings can be validated using the results of Bhadauria and Kiran in [4] as they have also claimed that "for a fixed value of other parameters, the critical Rayleigh number for the onset of oscillatory convection decreases with an increase in the value of λ_1", indicating that the effect of increasing viscoelastic parameter is to advance the onset of oscillatory convection. Thus, it is confirmed that the elastic behavior of the non-Newtonian fluids leads to oscillatory motions; hence, the heat transfer increases. Further, the effect of retardation parameter λ_2 is found to stabilize the system as the heat transfer decreases with increasing λ_2.

Further, the effects of ferromagnetic parameters M_1 and M_3 on the heat transfer rate Nu is depicted in Figs. 9 and 10, respectively. From the variations, it is observed that both M_1 and M_3 boost the heat transfer rate, see [10].

The supercritical pitchfork bifurcations for the ferromagnetic viscoelastic fluid layer are depicted in Figs. 11 and 13 under the effects of λ_1 and λ_2, respectively. Analogously, the pitchfork bifurcation phase plots, i.e., $B_1{}'(s)$ versus $B_1(s)$, are drawn in Figs. 12 and 14. The examination of the pitchfork bifurcation diagrams (Figs. 11 and 12) reveals that the equilibrium point is found to be unstable at $s = 0$ for each fixed value of the stress relaxation time (λ_1 =0.3,

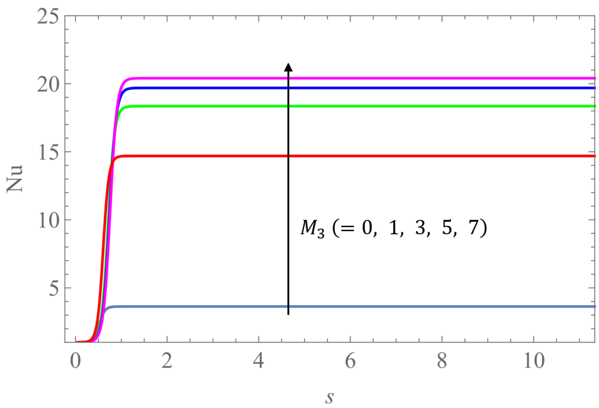

Fig. 10. Nu as a function of s for $\lambda_1 = 0.4$, $\lambda_2 = 0.1$, $\sigma = 10$ and $M_1 = 10$

Table 1. Values of the critical wave number a_c and critical Rayleigh number R_c for the *stationary case*

M_1	M_3	a_c	R_c^{Steady}
0	0	2.22144	657.511
0	1	2.22144	657.511
1	0	2.22144	657.511
1	1	2.51404	482.903
10	1	3.00248	129.390

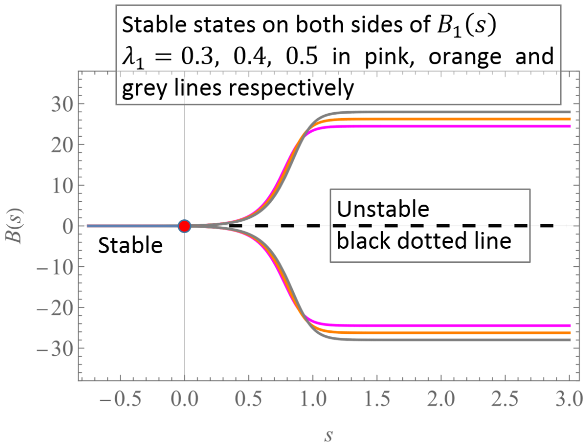

Fig. 11. Supercritical pitchfork bifurcation for $\lambda_2 = 0.1$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

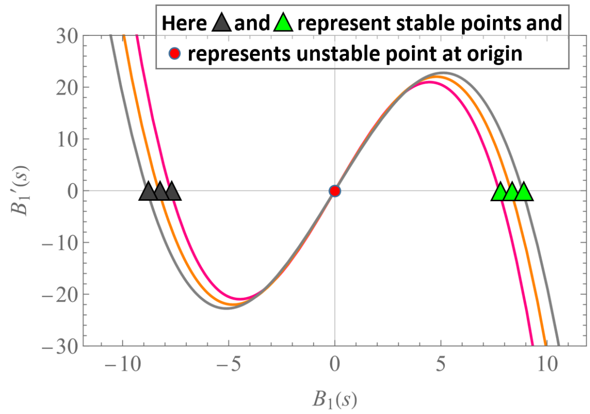

Fig. 12. Pitchfork bifurcation phase portraits for $\lambda_2 = 0.1$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

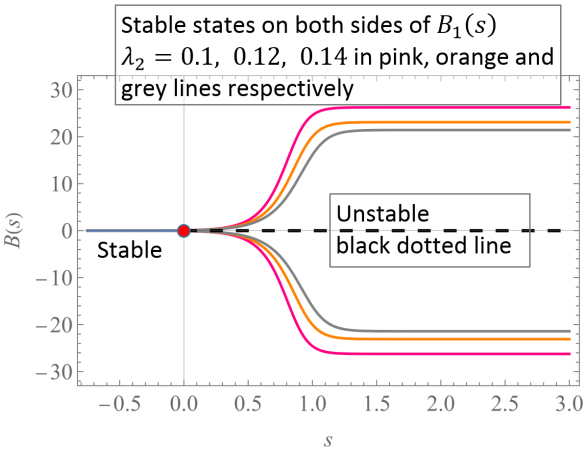

Fig. 13. Supercritical pitchfork bifurcation for $\lambda_1 = 0.4$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

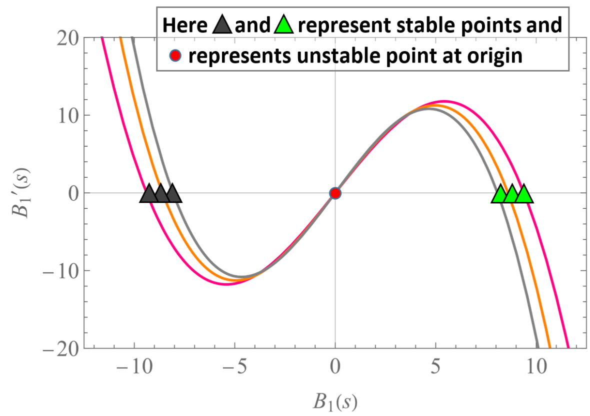

Fig. 14. Pitchfork bifurcation phase portraits for $\lambda_1 = 0.4$, $\sigma = 10$, $M_1 = 10$ and $M_3 = 5$

0.4, 0.5) and the value of amplitude grows with increasing stress relaxation time. Also, the analysis of the variations in Figs. 13 and 14 related to the pitchfork bifurcation shows that the equilibrium point is also unstable at $s = 0$ for every fixed value of the strain retardation time ($\lambda_2 = 0.1,\ 0.12,\ 0.14$). However, the value of amplitude declines with an increase in strain retardation time as compared to the stress relaxation time.

In Tables 1 and 2, the comparison of numerical values of R_c with regard to a_c for stationary and oscillatory cases is presented. It is seen that oscillations are not possible for an ordinary ferrofluid layer ($\lambda_1 = \lambda_2 = 0$) as the frequency has complex values.

In Tables 3 and 4, the critical values for ${R_c}^{\text{Steady}}$ and ${R_c}^{\text{Ov}}$ corresponding to the calculated values of a_c for steady and oscillatory cases, respectively, are presented and compared with the values published in [1,6,12,21], which are in a good agreement with the earlier obtained results.

5. Conclusions

In the present analysis, the effects of the Prandtl number σ, magnetic number M_1, measure of nonlinearity of magnetization M_3, and stress relaxation λ_1 and strain retardation λ_2 times on

Table 2. Values of the critical wave number a_c, critical Rayleigh number R_c, and oscillatory frequency ω for the *oscillatory case* with $\sigma = 10$

Fluid type	λ_1	λ_2	M_1	M_3	a_c	R_c^{Ov}	ω
Newtonian fluid	0	0	10	1	—	—	complex (no oscillations)
Maxwellian fluid	0.3	0	0	0	4.62938	33.0894	30.3502
			0	1	4.62938	33.0894	30.3502
			1	0	4.62938	33.0894	30.3502
			1	1	5.26873	19.3187	33.6464
			10	1	5.84160	3.95853	33.6639
Oldroydian fluid	0.3	0.1	0	0	2.48661	361.77	7.35643
			0	1	2.48661	361.77	7.35643
			1	0	2.48661	361.77	7.35643
			1	1	2.81918	255.538	7.94771
			10	1	3.30287	64.6892	8.85163

Table 3. Values of the critical wave number a_c and corresponding critical Rayleigh number R_c are compared for the linear *stationary case* with some limiting values of M_1 and M_3

Authors	$M_1 = M_3 = 0$, $\sigma = 1$		$M_1 = M_3 = 1$, $\sigma = 1$		$M_1 = 10$, $M_3 = 1.5$, $\sigma = 10$	
	R_c	a_c	R_c	a_c	R_c	a_c
Chandrasekhar [6]	657.511	2.22	—	—	—	—
Neha et al. [1]	657.511	2.221	482.903	2.51404	—	—
Finlayson [12]	—	—	482.903	2.5140	—	—
Melson et al. [21]	—	—	—	—	109.864	2.888
Current analysis	657.511	2.2214	482.903	2.51404	109.864	2.88804

Table 4. Values of the critical wave number a_c and corresponding critical Rayleigh number R_c are compared for the linear *oscillatory case* with $\sigma = 10$, $M_1 = 10$, $M_3 = 1.5$ and limiting values of λ_1, λ_2

Authors	Maxwellian fluid $\lambda_1 = 0.2$, $\lambda_2 = 0$		Oldroydian fluid $\lambda_1 = 0.2$, $\lambda_2 = 0.1$	
	R_c^{Ov}	a_c	R_c^{Ov}	a_c
Melson et al. [21]	6.7967	5.97931	85.61	3.18589
Current analysis	6.79671	5.97931	85.6101	3.18589

linear and weakly nonlinear ferroconvection have been studied.
For the linear stability case:

- λ_2 shows a stabilizing effect, whereas λ_1 shows a destabilizing effect on the onset of oscillatory instability; however, these parameters have no effect on the steady convection.
- $M_1$and M_3 both destabilize the onset of ferroconvection.

- Overstability is a preferred mode of instability.

For the weakly nonlinear stability case:

- The Nusselt number Nu decreases for increasing values of λ_2, while it grows with increasing values of λ_1.
- The rate of heat transfer declines with the increasing value of the Prandtl number.
- $M_1$and M_3 have a destabilizing effect on the nonlinear stability, as the heat is transferred more quickly in magnetic ferrofluids than in viscoelastic/ordinary fluids.
- The amplitude grows with increasing stress relaxation time; however, the amplitude decreases with increasing strain retardation time to reach stable positions.

References

[1] Aanam A, N., Siddheshwar, P. G., Nagouda, S. S., Pranesh, S., Thermoconvective instability in a vertically oscillating horizontal ferrofluid layer with variable viscosity, Heat Transfer 49 (8) (2020) 4543–4564. https://doi.org/10.1002/htj.21840

[2] Basu, R., Layek, G. C., The onset of thermo-convection in a horizontal viscoelastic fluid layer heated underneath, Thermal Energy Power Engineering 1 (2012) 1–9.

[3] Bhadauria, B. S., Hashim, I., Siddheshwar, P. G., Effect of internal heating on weakly non-linear stability analysis of Rayleigh-Bénard convection under g-jitter, International Journal of Non-Linear Mechanics 54 (2013) 35–42. https://doi.org/10.1016/j.ijnonlinmec.2013.03.001

[4] Bhadauria, B. S., Kiran, P., Weak nonlinear oscillatory convection in a viscoelastic fluid layer under gravity modulation, International Journal of Nonlinear Mechanics 65 (2014) 133–140. https://doi.org/10.1016/j.ijnonlinmec.2014.05.002

[5] Bhadauria, B. S., Singh, M. K., Singh, A., Singh, B. K., Kiran, P., Stability analysis and internal heating effect on oscillatory convection in a viscoelastic fluid saturated porous medium under gravity modulation, International Journal of Applied Mechanics and Engineering 21 (4) (2016) 785–803. https://doi.org/10.1515/ijame-2016-0046

[6] Chandrasekhar, S., Hydrodynamic and hydromagnetic stability, New York, Dover Publication, 1961.

[7] Dhiman, J. S., Patil, P. M., Sood, S., Modified stability analysis of double-diffusive convection in viscoelastic fluid layer saturating porous media, Heat Transfer 55 (2) (2022) 1497–1528. https://doi.org/10.1002/htj.22752

[8] Dhiman, J. S., Sharma, N., Effect of temperature dependent viscosity on thermal convection in ferrofluid layer, Journal of Theoretical and Applied Mechanics 51 (1) (2021) 3–21.

[9] Dhiman, J. S., Sharma, N., Thermal instability of hot ferrofluid layer with temperature-dependent viscosity, International Journal of Fluid Mechanics Research 45 (5) (2018) 389–398. https://doi.org/10.1615/InterJFluidMechRes.2018016842

[10] Dhiman, J. S., Sood, S., Linear and weakly non-linear stability analysis of oscillatory convection in rotating ferrofluid layer, Applied Mathematics and Computation 430 (2022) 1–16. https://doi.org/10.1016/j.amc.2022.127239

[11] Drazin, P. G., Reid, W. H., Hydrodynamic stability, Cambridge University Press, Cambridge, 1981.

[12] Finlayson, B. A., Convective instability of ferromagnetic fluids, Journal of Fluid Mechanics 40 (4) (1970) 753–767. https://doi.org/10.1017/S0022112070000423

[13] Kanchana, C., Siddheshwar, P. G., Pérez, L. M., Laroze, D., Comparison of the effect of suction-injection-combination on Rayleigh-Bénard convection in the case of asymmetric boundaries with those of symmetric ones, Physics of Fluids 35 (2023) 1–15. https://doi.org/10.1063/5.0146657

[14] Kanchana, C., Vélez, J. A., Pérez, L. M., Laroze, D., Siddheshwar, P. G., Influence of higher-order modes on ferroconvection, Chaos 32 (2022) 1–20. https://doi.org/10.1063/5.0097398

[15] Karaaslan, C. C., Bifurcation analysis and its applications, In: Numerical simulation – From theory to industry, M. Andriychuk (editors), IntechOpen, 2012. https://doi.org/10.5772/50075

[16] Kim, M. C., Lee, S. B., Kim, S., Chung, B. J., Thermal instability of viscoelastic fluids in porous media, International Journal of Heat and Mass Transfer 46 (26) (2003) 1565–1572. https://doi.org/10.1016/S0017-9310(03)00363-6

[17] Laroze, D., Martinez-Mardones, J., Pérez, L. M., Amplitude equation for stationary convection in a viscoelastic magnetic fluid, International Journal of Bifurcation and Chaos 20 (2) (2010) 235–242. https://doi.org/10.1142/S0218127410025673

[18] Laroze, D., Pérez, L. M., Bragard, J., Cordaro, E. G., Amplitude equation for stationary convection in a rotating viscoelastic magnetic fluid, Magnetohydrodynamics 47 (2) (2011) 1–7. https://doi.org/10.22364/mhd.47.2.7

[19] Laroze, D., Siddheshwar, P. G., Pleiner, H., Chaotic convection in a ferrofluid, Communications in Nonlinear Science Numerical Simulation 18 (9) (2013) 2436–2447. https://doi.org/10.1016/j.cnsns.2013.01.016

[20] Li, Z., Khayat, R. E., Finite-amplitude Rayleigh-Bénard convection and pattern selection for viscoelastic fluids, Journal of Fluid Mechanics 529 (2005) 221–251. https://doi.org/10.1017/S0022112005003563

[21] Melson, A. C., Siddheshwar, P. G., Sekhar, G. N., Nonlinear analysis of the effect of viscoelasticity on ferroconvection, Heat Transfer 50 (4) (2021) 3861–3878. https://doi.org/10.1002/htj.22055

[22] Rosenblat, S., Thermal convection in viscoelastic liquid, Journal of Non-Newtonian Fluid Mechanics 21 (2) (1986) 201–223. https://doi.org/10.1016/0377-0257(86)80036-2

[23] Rosensweig, R. E., Ferrohydrodynamics, Cambridge University Press, Cambridge, 1985.

[24] Siddheshwar, P. G., Oscillatory convection in a viscoelastic-Boussinesq-ferromagnetic fluid, Indian Journal of Engineering and Materials Sciences 5 (1998) 453–459.

[25] Siddheshwar, P. G., Oscillatory convection in viscoelastic ferromagnetic/dielectric liquids, International Journal of Modern Physics B 16 (17) (2002) 2629–2635. https://doi.org/10.1142/S0217979202012761

[26] Siddheshwar, P. G., Kanchana, C., Pérez, L. M., Laroze, D., Influence of symmetric/asymmetric boundaries on axisymmetric convection in a cylindrical enclosure in the presence of a weak vertical throughflow, Communications in Nonlinear Science Numerical Simulation 126 (2023) 1–16. https://doi.org/10.1016/j.cnsns.2023.107495

[27] Siddheshwar, P. G., Laroze, D., Kanchana, C., Lakshmi, K. M., Two-dimensional Rayleigh-Bénard convection of viscoelastic liquids in Cartesian and cylindrical coordinates: Regular and chaotic regimes, Europian Physical Journal Plus 137 (2022) 1–25. https://doi.org/10.1140/epjp/s13360-022-03095-7

[28] Siddheshwar, P. G., Sekhar, G. N., Jayalatha, G., Effect of time-periodic vertical oscillations of the Rayleigh-Bénard system on nonlinear convection in viscoelastic liquids, Journal of Non-Newtonian Fluid Mechanics 165 (19–20) (2010) 1412–1418. https://doi.org/10.1016/j.jnnfm.2010.07.008

[29] Siddheshwar, P. G., Sri Krishna, C. V., Unsteady non-linear convection in a second-order fluid, International Journal of Non-Linear Mechanics 37 (2) (2002) 321–330. https://doi.org/10.1016/S0020-7462(00)00119-0

[30] Siddheshwar, P. G., Sushma, T. S., Reduction of a tri-modal Lorenz model of ferrofluid convection to a cubic-quintic Ginzburg-Landau equation using the center manifold theorem, Differential Equations and Dynamical Systems (2021) 1–19. https://doi.org/10.1007/s12591-021-00565-9

[31] Siddheshwar, P. G., Suthar, O. P., Chinnaswamy, K., Finite-amplitude ferro-convection and electro-convection in a rotating Fluid, SN Applied Sciences 1 (2019) 1542–1553. https://doi.org/10.1007/s42452-019-1549-2