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# Thermodynamic effects of torsional surface waves in heterogeneous and homogeneous elastic semi-infinite substratum

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**Abstract**

The prime goal of the study is to investigate the relevance of torsional surface wave propagation in a thermally conducting heterogeneous (functionally graded) semi-infinite medium. The closed form of the dispersion equation for the propagation of torsional waves in heterogeneous elastic media has been established by assuming equations of motion in the temperature field with proper boundary conditions. One particular case where torsional vibration generates wave propagation in homogeneous materials has been addressed, and therefore, dispersion equation has been developed. The phase velocity and attenuation coefficient in both homogeneous and heterogeneous half-space have been computed to demonstrate the wave properties. In order to investigate the functionally graded parameter associated with the elastic medium, the effects of heat flow components related to thermodynamic forces, and the coupling coefficient related to the deformation and temperature fields, numerical calculations with graphical interpretation have been performed separately for phase velocity and attenuation coefficient.


*Keywords:* torsional surface waves, thermally conducting medium, phase velocity, attenuation coefficient, coupling coefficient

## 1. Introduction

Theoretical and quantitative modeling of elastodynamic problems is important in practically every branch of natural science and contemporary engineering. The elastodynamic problems for initially stressed structures are fascinating and of significant concerns involving nonlinear dynamical repercussions in elastic media. One of them is thermo-elasticity, which examines the relationship between the strain and temperature fields and, using irreversible thermodynamics, combines two distinct and independently established sciences: elasticity theory and heat conduction theory. Elastostatics and elastodynamics, the two core disciplines of elasticity theory, are often developed under different thermodynamic assumptions. Static problems are thought to be isothermal, whereas dynamic problems are thought to be adiabatic. The preceding two concepts are combined into a single whole by thermoelasticity. Thermoelasticity is currently a well-established field with specified fundamental assumptions, basic relations, and differential equations. Thus, the relevance of thermoelasticity stems mostly from its scientific qualities and extensions of other ideas.

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The propagation of surface waves in a heterogeneous layered medium has long aroused theoretical seismologists' interest. Surface wave propagation across the surface of elastic half-spaces that are homogeneous and heterogeneous is a recognized and significant feature of wave theory. Many well-known scientists have investigated the propagation of torsional surface waves in elastic media with varying degrees of heterogeneity. In 1968, Thomas [18] investigated the torsional wave propagation in an elastic half-space and explained the characteristics behavior of torsional waves. In 1975, the dynamic problems of thermoelasticity were studied by Nowacki [13]. In this article, the torsional surface waves propagation due to thermal conductivity acted as an external force. Similarly, Biot [2] analyzed the mechanics of incremental deformations of the elastic homogeneous and heterogeneous half-space. The forced torsional oscillations of an elastic half-space and an elastic stratum was studied by Collins [6]. Torsional surface waves in heterogeneous anisotropic half-space under initial stress was discussed by Chattopadhyay et al. [5]. Likewise, torsional surface waves in an initially stressed anisotropic porous medium was explored by Dey et al. [8]. In [7], Dey at al. have also contributed by developing ideas on the propagation of torsional waves in various inhomogeneous half-spaces. Recently, the propagation and attenuation of Rayleigh-type surface waves are investigated in the presence of imperfect bonding [15].

In 2014, the propagation of torsional waves in a viscoelastic layer over an inhomogeneous half-space was studied [11]. Similarly, the torsional waves propagation in an initially stressed dissipative cylinder was discussed by Selim in [16]. The propagation of torsional wave in a non-homogeneous crustal layer over a dry sandy mantle was discussed by Kundu et al. [12]. Dispersion and attenuation of a torsional wave in a viscoelastic layer bonded between a layer and a half-space of dry sandy media was investigated by Alam et al. [1]. The theories related to torsional surface waves in a gradient-elastic half-space were analyzed by Georgiadis et al. [9]. Torsional surface waves in inhomogeneous elastic media were scrutinized by Vardoulakis [19] in 1984. Likewise, the propagation of torsional surface waves in viscoelastic medium was studied by Dey et al. [7]. In 2015, Gourgiotis [10] has also established the Toupin-Mindlin gradient theory on torsional and horizontally polarized shear surface waves in an isotropic and homogenous elastic half-space. Singh et al. [17] investigated stresses and displacements induced due to a load moving with uniform velocity on the free rough surface of an irregular transversely isotropic functionally graded piezoelectric material (FGPM) substrate. The dynamic response of heterogeneity and reinforcement on the propagation of torsional surface waves was examined by Paswan et al. [14]. Similarly Chattaraj et al. [3,4], observed the propagation of torsional surface waves in an anisotropic poroelastic medium under initial stress as well as the dispersion of torsional surface waves in an anisotropic layer over porous half-space under gravity.

In this research, we investigate an elastic medium in which torsional surface waves propagate due to an external force in the form of heat conduction. We initiate this study with the understanding that the deformation of a body is always accompanied by changes in heat content and, consequently, body temperature. As a result, the internal energy of the body becomes dependent on factors related to deformation and temperature of the half-space. Therefore, we assume two fundamental equations: one involving displacement components, and the other related to heat components. The closed form of the dispersion equation is obtained by applying analytical methods with the necessary boundary conditions. For illustration, the problem is modeled by assuming the medium to be heterogeneous, and the dispersion equation for the propagation of torsional waves in a heterogeneous half-space is derived using a similar approach. The effects of heterogeneity factors, heat flow through the medium, and the coefficients governing deformation and temperature fields, along with the material properties, are presented graphically.

## 2. Framework of the problem

In order to analyze the problem, a cylindrical coordinate system is taken into account, where $r$, $\theta$ and $z$ axes represent radial, azimuthal and vertical directions, respectively. In our problem, the $z$-axis is directed positively towards the center of the Earth. The origin of the coordinate system is located at the surface of the half-space at the circular region. Due to heat flow components, the torsional surface waves are related to thermodynamic forces in a thermoelastic heterogeneous as well as homogeneous half-space. The study has been designed to get the dispersion equation of the torsional surface wave in an elastic medium due to thermal effect, as shown in the Fig. 1.

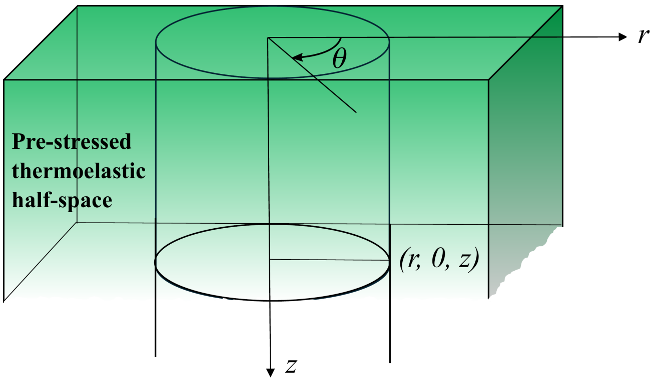


Fig. 1. Geometry of the problem

## 3. Solution of the problem

The dynamical equations of a cylindrical thermoelastic medium are given by [2]

$$\begin{aligned}\frac{\partial \tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r}&=\varrho\frac{\partial^2 u}{\partial t^2}+\gamma T,\\ \frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}&=\varrho\frac{\partial^2 v}{\partial t^2}+\gamma T,\\ \frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}+\frac{2\tau_{rz}}{r}&=\varrho\frac{\partial^2 w}{\partial t^2}+\gamma T,\end{aligned} \tag{1}$$

where the displacement components $u$, $v$ and $w$ are directed along the $r$, $\theta$ and $z$ directions, respectively. In addition, $\tau_{ij} = \tau_{ji}$ $(i, j = 1, 2, 3)$ are the elastic incremental components of stresses and $t$ and $\varrho$ denote the time and density of the substrate, respectively. $T$ represents the temperature increment measured from the reference state and $\gamma$ is the symbol for the free energy. Assuming that the torsional waves only involve circumferential displacement, so the displacement vectors along the radial, circumferential, and vertical directions are as follows:

$$u = 0, \quad v = v(r, z, t), \quad w = 0. \tag{2}$$

Now the stress-displacement relation for a thermoelastic gradient substrate can be written as

$$\begin{aligned}
\tau_{rr} &= \lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial r}, & \tau_{r\theta} &= \mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right),\\
\tau_{\theta\theta} &= \lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)+2\mu\left(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}\right), & \tau_{rz} &= \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),\\
\tau_{zz} &= \lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}, & \tau_{\theta z} &= \mu\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right).
\end{aligned} \tag{3}$$

where $\lambda$ and $\mu$ represent the Lame's constants. By implementing (2) and (3), we get

$$\tau_{rr}=\tau_{\theta\theta}=\tau_{zz}=\tau_{zr}=0, \quad \tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right), \quad \tau_{\theta z}=\mu\left(\frac{\partial v}{\partial z}\right). \tag{4}$$

It is to be noted that the substrate of the considered thermoelastic structure is functionally graded. In view of this, the functional gradient property associated with the rigidity and the density of the stratum may be written as

$$\mu=\mu_0 e^{az}, \qquad \varrho=\varrho_0 e^{az}, \tag{5}$$

where $\mu_0$ and $\varrho_0$ are the stiffness and density of the materials of the medium at $z=0$, respectively. The functionally gradient parameters related to the substratum are denoted by $a$.

By applying (2), (4) and (5), equation (1) can be converted into the following form:

$$\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 v}{\partial z^2}+a\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}=\frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2}+2\alpha_T T, \tag{6}$$

where $c_2=\sqrt{\mu_0/\varrho_0}$ is the shear velocity, and $\alpha_T=\gamma/(2\mu)$ is the linear thermal expansion coefficient. Equation (6) represents a wave equation with cylindrical symmetry, involving an additional term, temperature $T$, which accounts for the inclusion of damping or heat conduction effects. Such equation is commonly encountered in the context of wave propagation in cylindrical structures under the influence of external factors. Specifically, the presence of the term $\left(\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)$ is characteristic of the cylindrical coordinate system, leading to solutions that naturally involve Bessel functions. The use of Bessel functions for solving wave equations with cylindrical symmetry has been widely adopted in the literature [13].

The solution of (6), when torsional surface wave propagates along the radial direction with an amplitude of displacement as a function of depth, may be taken in view of [7] as

$$\begin{aligned}
v(r,z,t) &= V(z)J_1(kr)e^{i\omega t},\\
T(r,z,t) &= \Theta(z)J_1(kr)e^{i\omega t},
\end{aligned} \tag{7}$$

where $\omega=kc$ represents the angular frequency. The phase velocity and angular wave number of torsional surface waves are denoted by $c$ and $k$, respectively. $J_1(kr)$ denotes the Bessel function of the first kind. By substituting (7) in (6), the displacement equation converts to the form

$$\frac{d^2 V}{dz^2}+a\frac{dV}{dz}-k^2\left(1-\frac{c^2}{c_2^2}\right)V=2\alpha_T\Theta. \tag{8}$$

Last but not the least, the dynamical equation of heat conduction may be taken as

$$\nabla^2 T-\frac{1}{\chi}T-\eta\nabla^2 v=0, \tag{9}$$

where $T$ is the temperature increment measured from the reference state, $\chi$ is the thermal diffusivity, and $\eta$ is the thermal conductivity.

By considering the Bessel function of the first kind as the solution of (9) and with the help of (7), equation (9) reduces to

$$\frac{d^2\Theta}{dz^2} - \left(k^2 + \frac{i\omega}{\chi}\right)\Theta - i\omega\eta\left[\frac{d^2V}{dz^2} + a\frac{dV}{dz} - k^2V\right] = 0. \tag{10}$$

Now, in order to obtain $T$, we get from (8)

$$\Theta = \frac{1}{2\alpha_T}\left[\frac{d^2V}{dz^2} + \frac{dV}{dz} - k^2\left(1 - \frac{c^2}{c_2^2}\right)V\right]. \tag{11}$$

Then, making use of (11) in (10), we may obtain the following fourth order differential equation:

$$\begin{aligned}\frac{d^4V}{dz^4} + a\frac{d^3V}{dz^3} - \left[2k^2 - \sigma^2 + q(1+\varepsilon)\right]\frac{d^2V}{dz^2} - a\left[k^2 + q(1+\varepsilon)\right]\frac{dV}{dz} \\ + \left[\left(k^2 + \frac{i\omega}{\chi}\right)\left(k^2 - \frac{\omega^2}{c_2^2}\right) + k^2 q\varepsilon\right]V = 0,\end{aligned} \tag{12}$$

where $\sigma = \omega/c_2$, $q = i\omega/\chi$, and $\varepsilon = 2\eta\chi\alpha_T$ is regarded as the coupling coefficient.

Let us assume the solution of (12) as

$$V(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}, \tag{13}$$

where $C_1$ and $C_2$ are arbitrary constants. The mathematical expressions for $\lambda_1$ and $\lambda_2$, which are the roots of the auxiliary equation (13), are provided in Appendix. Furthermore, we may have

$$v(r, z, t) = \left(C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}\right) J_1(kr) e^{i\omega t}. \tag{14}$$

Finally, using (13) in (11), $T$ can be obtained in view of the second equation of (10) as

$$\begin{aligned}T = \frac{1}{2\alpha_T}\Bigg\{C_1\left[\lambda_1^2 + a\lambda_1 - \left(k^2 - \frac{\omega^2}{c_2^2}\right)\right]e^{\lambda_1 z} \\ + C_2\left[\lambda_2^2 + a\lambda_2 - \left(k^2 - \frac{\omega^2}{c_2^2}\right)\right]e^{\lambda_2 z}\Bigg\} J_1(kr) e^{i\omega t}.\end{aligned} \tag{15}$$

This is the required heat expression for the propagation of a torsional surface wave in a heterogeneous thermo-elastic half-space.

## 4. Boundary conditions

For the model considered in the present study, the appropriate boundary conditions are:

(i) $\frac{dv}{dz} = 0$ at $z = 0$, i.e., the medium is stress-free at the boundary.

(ii) $\frac{dT}{dz} = 0$ at $z = 0$, i.e., the temperature gradient on elastic half-space is assumed to be zero at this boundary.

Using boundary conditions (i) and (ii), we get

$$C_1\lambda_1 e^{(\lambda_1 z + i\omega t)} + C_2\lambda_2 e^{(\lambda_2 z + i\omega t)} = 0 \tag{16}$$

and

$$C_1\lambda_1\left[\lambda_1^2+a\lambda_1-\left(k^2-\frac{\omega^2}{c_2^2}\right)\right]e^{(\lambda_1 z+i\omega t)}+C_2\lambda_2\left[\lambda_2^2+a\lambda_2-\left(k^2-\frac{\omega^2}{c_2^2}\right)\right]e^{(\lambda_2 z+i\omega t)}=0. \tag{17}$$

For a non-trivial solution, eliminating $C_1$ and $C_2$ from (16) and (17), we obtain

$$\lambda_1\lambda_2\left[\lambda_1^2+a\lambda_1-\left(k^2-\frac{\omega^2}{c_2^2}\right)\right]-\lambda_1\lambda_2\left[\lambda_2^2+a\lambda_2-\left(k^2-\frac{\omega^2}{c_2^2}\right)\right]=0. \tag{18}$$

Further it is derived from (18) as

$$\lambda_1+\lambda_2+a=0. \tag{19}$$

By substituting the expressions of $\lambda_1$ and $\lambda_2$ in (19), we get

$$2^{\frac{2}{3}}\xi_1^2+4\left[2k^2+4\left\{q(1+\varepsilon)-\sigma^2\right\}\right]J_1+2\xi_2=0. \tag{20}$$

This is the general dispersion equation for a thermo-elastic heterogeneous half-space. The mathematical expressions for $\xi_1$ and $\xi_2$ are provided in Appendix. It is also worth noting that equation (20) is for a complex quantity. By separating its real and imaginary components, the phase velocity and the attenuation coefficient are obtained, respectively. These values are computed numerically and analyzed in detail.

*Case 1:*

For $a=0$, we obtain the homogeneous case and the dispersion equation will be

$$\left[2-\frac{c^2}{c_2^2}+\frac{ic}{k\chi}(1+\varepsilon)\right]^2=4\left[\left(1+\frac{ic}{k\chi}\right)\left(1-\frac{c^2}{c_2^2}\right)-\frac{ic\varepsilon}{k\chi}\right]. \tag{21}$$

This is the necessary dispersion equation of the torsional surface waves propagation in a thermoelastic homogeneous half-space.

*Case 2:*

When $a=0$ and the elements associated with the heat conduction are removed, equation (21) reduces to

$$\left(2-\frac{c^2}{c_2^2}\right)^2=4\left(1-\frac{c^2}{c_2^2}\right), \tag{22}$$

and it results in $c=0$. This leads to the conclusion that torsional surface waves do not propagate in a homogeneous elastic half-space.

## 5. Numerical computation and discussion

In order to evaluate the influenced characteristics of torsional wave propagation due to various parameters such as the heterogeneity parameter $a/k$, thermal coefficient $q$ and coupling coefficient $\varepsilon$ of four different materials (copper, steel, aluminium, and lead), theoretical observations have been carried out numerically and graphically. These results have been used to plot graphs, indicating the variations of phase velocity $c/c_2$ and attenuation coefficients $\delta$ for non-homogeneous (Figs. 2–4) and homogeneous (Figs. 5 and 6) cases. The coupling coefficient

Table 1. Coupling coefficient $\varepsilon$ for four different materials [13]

| Medium | Coupling coefficient $\varepsilon$ |
|---|---|
| aluminum (Al) | $3.56 \times 10^{-2}$ |
| copper (Cu) | $1.68 \times 10^{-2}$ |
| steel (Fe) | $2.97 \times 10^{-2}$ |
| lead (Pb) | $7.33 \times 10^{-2}$ |

values of the distinct material are listed in Table 1. Unless otherwise stated, the following values of the parameters are considered:

$$\frac{a}{k} = 0.0, 0.2, 0.4, 0.6, 0.8, \qquad q = 0.00, 0.05, 0.10, 0.15, 0.20.$$

Figs. 2a and 2b illustrate the influence of the power gradient parameter $a/k$ associated with the functionally graded half-space on the phase velocity $c/c_2$ and the attenuation coefficient $\delta$ of the torsional surface waves, respectively. It is clear from these figures that both the phase velocity and the attenuation coefficient dicrease monotonically with the rise in the wave number $k$. It is perceived that as the power gradient parameter $a/k$ increases, the phase velocity and the attenuation of the torsional waves decreases. From these figures, one can conclude that the gradient parameter has a substantial influence on the phase velocity and the attenuation of the torsional waves. Distinct curves of these figures unravel that the effect of the gradient parameter is more on the lower side of the wave number for the phase velocity, however, the effect of the same is more on the attenuation coefficient for the higher range of the wave number.

The impact of the coupling coefficient $\varepsilon$ on the phase velocity and the attenuation coefficient resulting from the propagation of torsional surface waves is shown in Figs. 3a and 3b, respectively. It is clear that when the wave number increases, both the phase velocity and the attenuation coefficient exhibit a decreasing trend. It is observed that the effect of the coupling coefficient on both the phase velocity and the attenuation is the lowest for the material Cu and

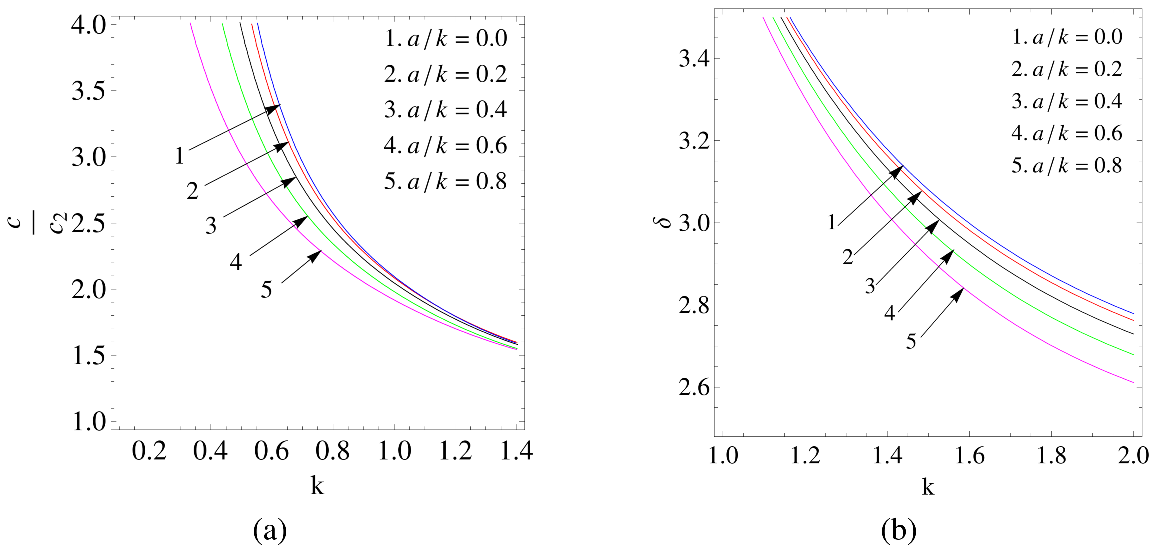


Fig. 2. Variation of (a) phase velocity and (b) attenuation coefficient against the wave number for the gradient parameter $a/k$ associated with the functionally graded semi-infinite medium

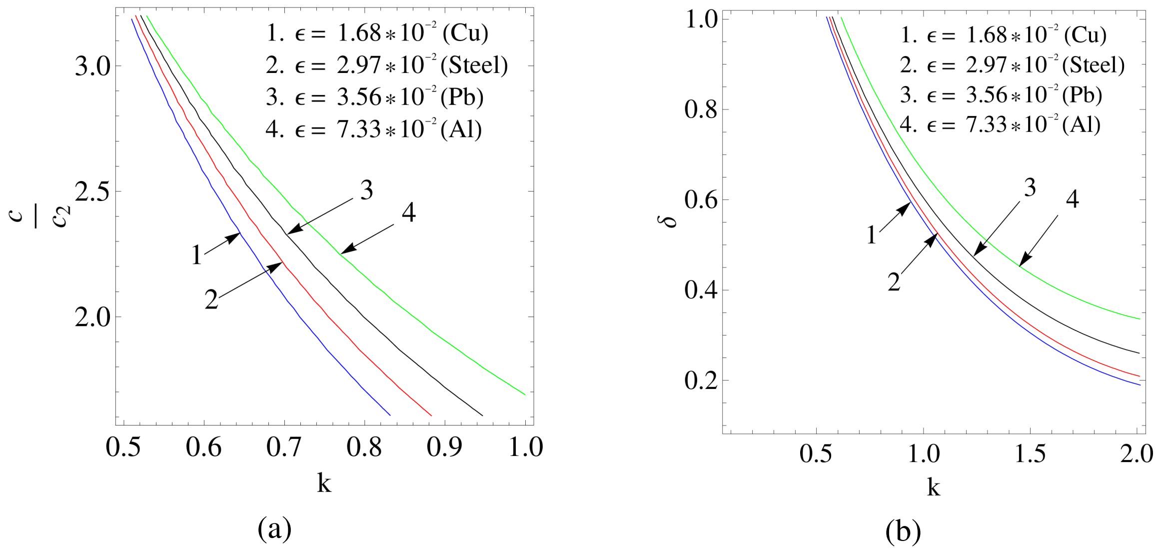


Fig. 3. Variation of (a) phase velocity and (b) attenuation coefficient against the wave number for the coupling coefficient $\varepsilon$ associated with the functionally graded semi-infinite medium

is the highest for the material Pb.

The impact of the thermal conducting parameter $q$ corresponding to the selected four functionally graded materials on the phase velocity and the attenuation coefficient is shown in Figs. 4a and 4b, respectively. It can be observed that in the absence of the thermal coefficient, the phase velocity as well as the attenuation coefficient attain the lowest value. With an increase of the thermal coefficient in the graded material, both physical phenomena get increased significantly.

The effect of the coupling coefficient on the phase velocity and the attenuation coefficient resulting from torsional surface waves for the homogeneous elastic half-space is shown in Figs. 5a and 5b, respectively. From these figures, it can be deduced that as the wave number increases, the phase velocity and the attenuation coefficient decrease. Furthermore, the study of phase ve-

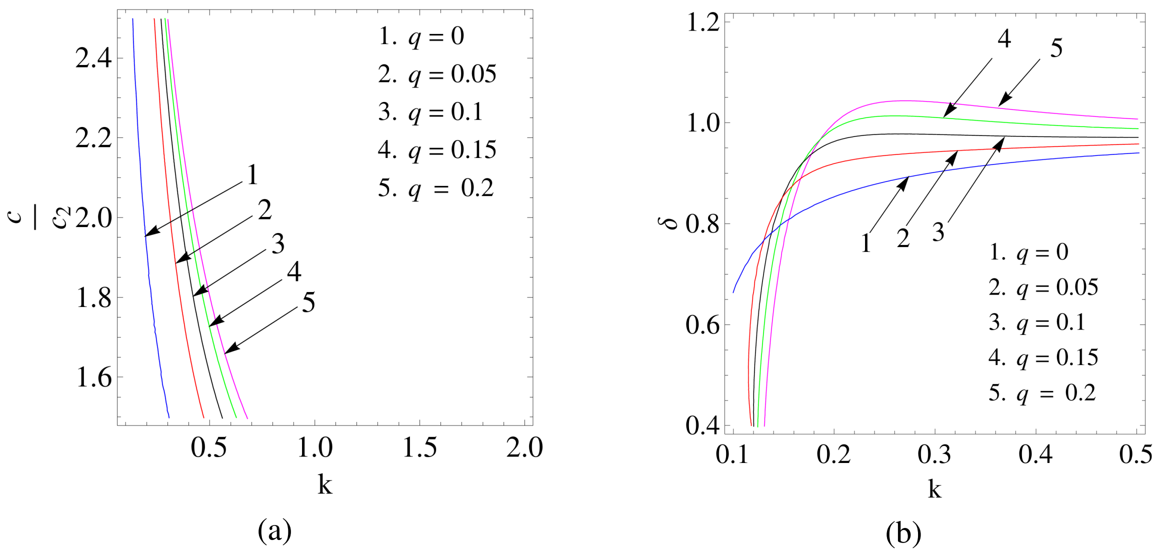


Fig. 4. Variation of (a) phase velocity and (b) attenuation coefficient against the wave number for the thermally conducting parameter $q$ associated with the functionally graded semi-infinite medium

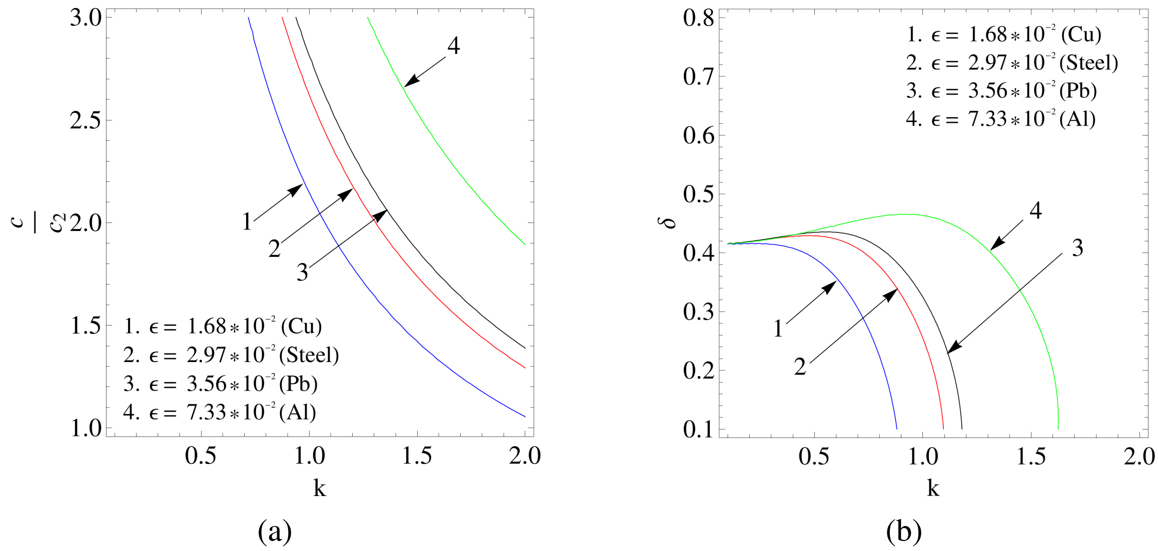


Fig. 5. Variation of (a) phase velocity and (b) attenuation coefficient against the wave number for the coupling coefficient $\varepsilon$ associated with the homogeneous semi-infinite medium

locity and attenuation curves reveals that these values are found to be maximal for the material Pb, while these are the lowest for the material Cu.

The impact of the parameter $q$ in an elastic homogeneous case is illustrated by means of Figs. 6a and 6b for the phase velocity and the attenuation coefficient, respectively. The values of $q$ have been assumed positive because torsional waves cannot travel through a homogeneous elastic semi-infinite substratum in the absence of a thermal coefficient. It has been observed that the phase velocity curves of torsional waves increase when the wave number increases, while the attenuation curves get decreased with the increase in the parameter $q$. The analysis of the phase velocity curves in these figures suggests that a similar trend is obtained for the functionally graded and homogeneous cases. By contrast, the attenuation coefficient has a contradicting nature in the case of functionally graded and homogeneous cases.

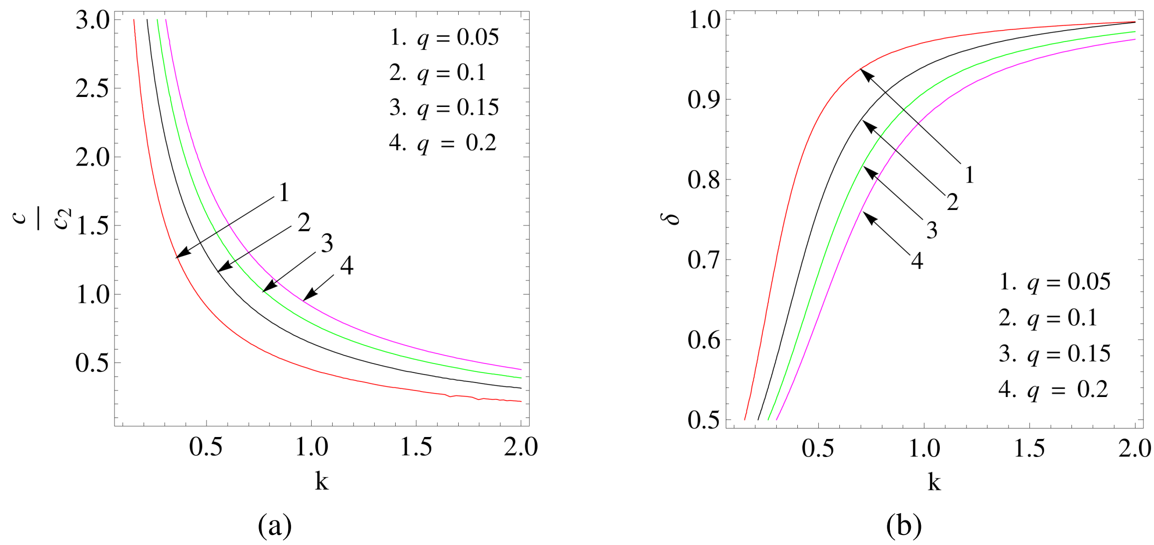


Fig. 6. Variation of (a) phase velocity and (b) attenuation coefficient against the wave number for the thermally conducting parameter $q$ associated with the homogeneous semi-infinite medium

## 6. Conclusions

In the present article, the propagation behavior of torsional surface waves in a thermally conducting, functionally graded, and homogeneous elastic substratum is investigated. Using an analytical method, the dispersion equation is derived for both graded and homogeneous half-spaces under suitable boundary conditions. A significant effect of the affecting parameters viz. power gradient parameter, thermal coefficient, heat flow, and coupling coefficient of the materials, on the phase velocity and the attenuation coefficient are examined. Both the phase velocity and the attenuation coefficient exhibit a decreasing trend with an increase in the wave number. By comparing the effect of the coupling coefficient across materials, for both homogeneous and heterogeneous cases, the phase velocity and the attenuation coefficient increase significantly.

The thermal conductivity parameter significantly influences the phase velocity and the attenuation coefficient in the torsional vibration dynamics for both homogeneous and heterogeneous cases. It is further established that the thermal conductivity plays a vital role in both cases, leading to notable variations in the phase velocity and the attenuation coefficient. Another key conclusion is that in the absence of thermal conductivity, the phase velocity and the attenuation coefficient show negligible behavior, indicating a lack of wave propagation.

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## Appendix

The expressions for $\lambda_1$, $\lambda_2$ appearing in (13) are as follows:

$$\lambda_1 = -\frac{a}{4} - \frac{1}{2}\sqrt{L_1} - \frac{1}{2}\sqrt{L_2}, \qquad \lambda_2 = -\frac{a}{4} - \frac{1}{2}\sqrt{L_1} + \frac{1}{2}\sqrt{L_2},$$

where

$$\begin{aligned}
L_1 &= \frac{a^2}{4} + 2k^2 + q + q\varepsilon - \sigma^2 + \frac{1}{3}\left(-2k^2 - q - q\varepsilon + \sigma^2\right) + \frac{\xi_2}{3\xi_1} + \frac{\xi_1}{32^{\frac{1}{3}}}, \\
L_2 &= \frac{a^2}{2} + 2k^2 + q + q\varepsilon - \sigma^2 + \frac{1}{3}\left(2k^2 + q + q\varepsilon - \sigma^2\right) - \frac{\xi_2}{3\xi_1} - \frac{\xi_1}{32^{\frac{1}{3}}} \\
&\quad - \frac{1}{4\sqrt{L_1}}\left[-a^3 + 8a\left(k^2 + q + q\varepsilon\right) + 4a\left(-2k^2 - q - q\varepsilon + \sigma^2\right)\right], \\
\xi_1 &= \Big\langle 27a^2\left(k^2 + q + q\varepsilon\right)^2 + 9a^2\left(k^2 + q + q\varepsilon\right)\left(-2k^2 - q - q\varepsilon + \sigma^2\right) \\
&\quad + 2\left(-2k^2 - q - q\varepsilon + \sigma^2\right)^3 + 27a^2\left(k^4 + k^2 q + k^2 q\varepsilon - k^2\sigma^2 - q\sigma^2\right) \\
&\quad - 72\left(-2k^2 - q - q\varepsilon + \sigma^2\right)\left(k^4 + k^2 q + k^2 q\varepsilon - k^2\sigma^2 - q\sigma^2\right) \\
&\quad + \Big\{-4\Big[3a^2\left(k^2 + q + q\varepsilon\right) + \left(-2k^2 - q - q\varepsilon + \sigma^2\right)^2
\end{aligned}$$

$$+ 12\left(k^4 + k^2q + k^2q\varepsilon - k^2\sigma^2 - q\sigma^2\right)\Big]^3$$
$$+ \left[27a^2\left(k^2 + q + q\varepsilon\right)^2 + 9a^2\left(k^2 + q + q\varepsilon\right)\left(-2k^2 - q - q\varepsilon + \sigma^2\right)\right.$$
$$+ 2\left(-2k^2 - q - q\varepsilon + \sigma^2\right)^3 + 27a^2\left(k^4 + k^2q + k^2q\varepsilon - k^2\sigma^2 - q\sigma^2\right)$$
$$\left.-72\left(-2k^2 - q - q\varepsilon + \sigma^2\right)\left(k^4 + k^2q + k^2q\varepsilon - k^2\sigma^2 - q\sigma^2\right)\right]^2\Big\}^{\frac{1}{2}}\Big\rangle^{\frac{1}{3}},$$
$$\xi_2 = 2^{\frac{1}{3}}\left[3a^2\left(k^2 + q + q\varepsilon\right) + \left(-2k^2 - q - q\varepsilon + \sigma^2\right)^2 + 12\left(k^4 + k^2q + k^2q\varepsilon - k^2\sigma^2 - q\sigma^2\right)\right].$$