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# Thin viscoelastic disc subjected to radial non-stationary loading

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**Abstract**

The investigation of non-stationary wave phenomena in isotropic viscoelastic solids using analytical approaches is the aim of this paper. Concretely, the problem of a thin homogeneous disc subjected to radial pressure load nonzero on the part of its rim is solved. The external excitation is described by the Heaviside function in time, so the non-stationary state of stress is induced in the disc. Dissipative material behaviour of solid studied is represented by the discrete material model of standard linear viscoelastic solid in the Zener configuration. After the derivation of motion equations final form, the method of integral transforms in combination with the Fourier method is used for finding the problem solution. The solving process results in the derivation of integral transforms of radial and circumferential displacement components. Finally, the type of derived functions singularities and possible methods for their inverse Laplace transform are mentioned.


*Keywords:* thin disc, radial loading, non-stationary state of stress, viscoelasticity, analytical solution, Laplace transform

## 1. Introduction

The analysis of thin circular solids vibrations is an important subject for many engineering applications, e.g. radial impact on wheels, transverse loading of disc brakes, dynamic loadings of disc drives, radial or transverse excitation of circular saw blades, impacts of meteorites on spinning satellites, impacts of soft bodies on rotating engine discs and blades etc. One can find lots of works dealing mainly with the transverse vibrations of circular or annular plates and with the in-plane vibrations of stationary or rotating discs.

The work [7] lies within the first mentioned group. The author concerns the analytical solution of static and dynamic problem of a transverse excited circular elastic plate. The Mindlin's plate theory is used and unlike the most of other works, the asymmetric problem is solved. The paper [8] from the same author treats the wave propagation in a thin elastic or viscoelastic layer induced by a transverse non-stationary loading. In this work, the author uses four different numerical or analytical models to describe the wave phenomena in a thin gelatin plate produced by prescribed pressure or velocity on its boundary. All four approaches are compared and their limitations are discussed. Another work concerning an impact problem of circular plates is [9]. The collective of authors treats the axisymmetric problem of transverse impact on an isotropic rotating disc analytically. Comparing results obtained for different angular velocities and for different materials, the role of centrifugal stresses in absorbing impact energy is discussed. Besides mentioned works, there exist lots of papers treating the stability problems of stationary or

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rotating discs. A large number of appropriate references can be found e.g. in [12], where the authors investigated the instability of travelling waves in the transverse vibration of a stationary disc induced by a rotating friction system.

The presented paper falls within the second set of problems, i.e. in-plane loaded stationary or rotating discs. This set of problems is not so numerous as the previous one, but number of works can be found here as well. From the older works, we can mention for example papers [3, 14] and [11], concerning in-plane stationary vibrations of elastic discs and annular rings produced by different types of dynamic loading. Another work dealing with the analytical solution of similar problem is [6]. Chen and Jhu applied Fourier-Bessel series in obtaining plane stress and displacement distributions in a rotating annular elastic disc under stationary edge load and studied these distributions approaching the critical value of angular velocity. Koh et al. in [10] solve the problem of rotating disc subjected to stationary load and the problem of stationary disc subjected to rotating load numerically. They present the new numerical method called Moving Element Method (MEM) and compare numerical results with the analytical solution in terms of complex Fourier-Hankel series. The advantages of the proposed numerical method over the finite element method are discussed.

Other works dealing with the stationary and mainly non-stationary problems of in-plane loaded thin elastic discs from the analytical point of view have origin in the Institute of Thermomechanics of the Czech Academy of Sciences. In [4] the authors briefly treat the one-dimensional (1D) problem of torsional impact on a thin elastic disc and the problem of radial impact on this disc in 1D and 2D. The last mentioned problem of plane stress is solved in [5] in detail and the functions describing the distributions of displacement components are derived.

In this paper, we utilize the experiences acquired by the solving of other non-stationary problems, namely the axisymmetric problem of thin viscoelastic plate vibrations induced by transverse loading [1] and the problem of thin viscoelastic Timoshenko beam vibrations [2], to make the generalization of the previously mentioned work [5] in the sense of material behaviour of the disc studied. Concretely, the material properties will be supposed viscoelastic and the standard linear viscoelastic model will be used for their description. The main purpose of this effort is to obtain new analytical results which can be then utilized for testing and verification of existing and new numerical methods. Contrary to the last mentioned work we will consider zero initial velocity of the disc because non-zero disc velocity does not influence the state of stress in the disc and cause predictable response in disc displacement.

In the first following part of this work, the mathematical model of the problem solved will be derived. Then the method used for solving the final system of motion equations will be described and the Laplace transforms of required displacement components will be presented. Finally, the type of singularities and the possibilities of the inverse Laplace transform will be briefly discussed.

## 2. The problem description and initial assumptions

The disc, in which the non-stationary state of stress is investigated, has constant unit thickness and finite radius $r_1$. This solid is subjected to the pressure radial loading that has constant amplitude $\sigma_0$ nonzero only on the part of disc rim specified by the angle $\alpha_0$ (see Fig. 1). The amplitude $\sigma_0$ changes according to the Heaviside function in time, so it causes non-stationary wave phenomena in the disc studied. Based on previous assumptions, the external excitation can be expressed by the function

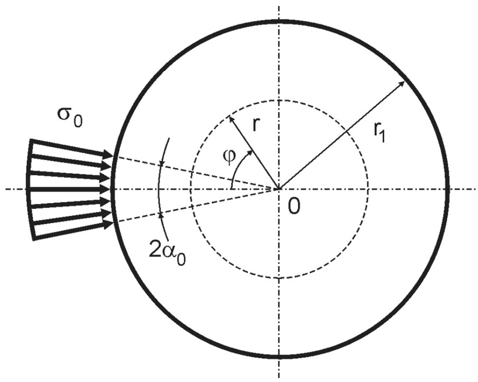


Fig. 1. The geometry of solved problem

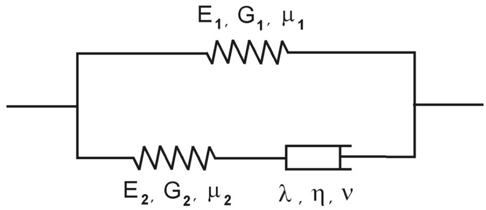


Fig. 2. The scheme of the Zener material model

$$\sigma_r(r,\varphi,t)|_{r=r_1} = \begin{cases} \sigma_0 H(t) & \text{for } \varphi \in \langle -\alpha_0, \alpha_0 \rangle, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $r$ and $\varphi$ denote polar coordinates in which the problem will be solved. Their orientation is obvious from Fig. 1.

The disc material will be supposed homogeneous and isotropic with viscoelastic behaviour. The dissipative material character will be represented by the discrete model of the standard linear viscoelastic solid. Namely, the Zener model, i.e. the Maxwell element with elastic spring in parallel, will be used (see Fig. 2).

With respect to the problem description, the state of plane stress occurs in the disc. This state is described by the nonzero radial $\sigma_r\,(r,\varphi,t)$, circumferential $\sigma_\varphi\,(r,\varphi,t)$ and shear $\tau_{r\varphi}\,(r,\varphi,t)$ stress components which all depend on $r$, $\varphi$ and $t$. Additionally, the corresponding strain components $\varepsilon_r\,(r,\varphi,t)$, $\varepsilon_\varphi\,(r,\varphi,t)$ and $\gamma_{r\varphi}\,(r,\varphi,t)$ and the displacement components $u_r\,(r,\varphi,t)$ and $u_\varphi\,(r,\varphi,t)$ make the definition of the state in arbitrary point of the disc complete.[1] The fourth nonzero strain component that represents strain in the direction perpendicular to the disc plane can be expressed using components $\varepsilon_r$ and $\varepsilon_\varphi$.

## 3. The derivation of mathematical model

### 3.1. Governing equations

Differential equations of equilibrium for a disc element can be easily derived from the system of motion equations (momentum conservation) for an three-dimensional continuum element [15]. Introducing corresponding inertial forces acting on a disc element and taking into account that there exist only three nonzero stress components and two nonzero displacement components, three motion equations reduce to two non-trivial equations in polar coordinates having the form

$$\rho\,\frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\,\frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{1}{r}\,\sigma_r - \frac{1}{r}\,\sigma_\varphi\,, \quad \rho\,\frac{\partial^2 u_\varphi}{\partial t^2} = \frac{\partial \tau_{r\varphi}}{\partial r} + \frac{1}{r}\,\frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{2}{r}\,\tau_{r\varphi}\,, \tag{2}$$

where $\rho$ denotes mass density.

[1] The functions notations without independent variables will be used in the following to make equations and relations more transparent, if possible with respect to mathematical explicitness.

Secondly, strain-displacement equations need to be specified to derive mathematical model of the problem solved. These relations can be formulated using strain-displacement equations for small displacements in cylindrical coordinates. With respect to the fact that all strain and displacement components of the disc are the functions of variables $r$, $\varphi$ and $t$, the required relations between strain and displacement components can be written as

$$\varepsilon_r = \frac{\partial u_r}{\partial r}\,, \quad \varepsilon_\varphi = \frac{1}{r}\,u_r + \frac{1}{r}\,\frac{\partial u_\varphi}{\partial \varphi}\,, \quad \gamma_{r\varphi} = \frac{1}{r}\,\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{1}{r}\,u_\varphi\,. \tag{3}$$

As mentioned above, the Zener model was chosen for the representation of viscoelastic disc properties. Owing to the linearity of the problem, constitutive equations can be derived using the principle of stress or strain superposition in the case of elements in parallel or in series, respectively. The corresponding relations can be found e.g. in [13] and for the state of plane stress and for zero initial stress and strain conditions there hold

$$\begin{aligned}
\sigma_r &= \frac{E_1\,(\varepsilon_r + \mu_1\varepsilon_\varphi)}{1-{\mu_1}^2} + \frac{E_2\,(\varepsilon_r + \mu_2\varepsilon_\varphi)}{1-{\mu_2}^2} - \frac{{E_2}^2}{\lambda\,(1-{\mu_2}^2)}\int_0^t (\varepsilon_r + \mu_2\varepsilon_\varphi)\,\mathrm{e}^{-\frac{E_2(t-\tau)}{\lambda}}\,\mathrm{d}\tau\,,\\
\sigma_\varphi &= \frac{E_1\,(\varepsilon_\varphi + \mu_1\varepsilon_r)}{1-{\mu_1}^2} + \frac{E_2\,(\varepsilon_\varphi + \mu_2\varepsilon_r)}{1-{\mu_2}^2} - \frac{{E_2}^2}{\lambda\,(1-{\mu_2}^2)}\int_0^t (\varepsilon_\varphi + \mu_2\varepsilon_r)\,\mathrm{e}^{-\frac{E_2(t-\tau)}{\lambda}}\,\mathrm{d}\tau\,,\\
\tau_{r\varphi} &= (G_1+G_2)\,\gamma_{r\varphi} - \frac{{G_2}^2}{\eta}\int_0^t \gamma_{r\varphi}\,\mathrm{e}^{-\frac{G_2(t-\tau)}{\eta}}\,\mathrm{d}\tau\,.
\end{aligned} \tag{4}$$

Material constants $E_i$, $G_i$ and $\mu_i$ are Young modulus, shear modulus and Poisson ratio of the alone standing spring ($i = 1$) or of the spring in the Maxwell element ($i = 2$), respectively (see Fig. 2). Symbols $\lambda$ and $\eta$ denote coefficients of normal and shear viscosities. Relations (4) were derived under the assumption of viscous and elastic Poisson ratios equality in the Maxwell model, i.e $\nu = \mu_2$.

### *3.2. The resulting form of motion equations*

Based on governing equations mentioned previously, we derive the resulting form of motion equations in this paragraph. For this purpose, we introduce constants $c_{2i}$ and $c_{3i}$ for $i = 1, 2$, which are analogous to the phase velocity of rotational (shear) waves in general continuum and to the phase velocity of dilatational waves in two-dimensional continuum, by relations

$$c_{2i} = \sqrt{\frac{G_i}{\rho}}\,, \quad c_{3i} = \sqrt{\frac{E_i}{\rho\,(1-\mu_i^2)}} \tag{5}$$

and further we define coefficients $\alpha$ and $\beta$ as the reciprocal values of relaxation times, i.e.

$$\alpha = \frac{E_2}{\lambda}\,, \quad \beta = \frac{G_2}{\eta}\,. \tag{6}$$

Introducing strain-displacement equations (3) and constitutive relations (4) into motion equations (2) using (5) and (6) and after some rearrangements, we obtain the final system of equations describing non-stationary wave phenomena in the studied viscoelastic disc:

$$\begin{aligned}
\frac{\partial^2 u_r}{\partial t^2} &= -2\beta\, c_{22}^2 \frac{1}{r}\int_0^t \left(\frac{\partial^2 u_\varphi}{\partial r \partial \varphi} - \frac{\partial \omega_z}{\partial \varphi}\right) \mathrm{e}^{-\beta\,(t-\tau)} \mathrm{d}\tau + \left(c_{31}^2 + c_{32}^2\right) \frac{\partial \Delta_d}{\partial r} - \\
&\quad -2\left(c_{21}^2 + c_{22}^2\right) \frac{1}{r}\frac{\partial \omega_z}{\partial \varphi} + \alpha \int_0^t \left(2c_{22}^2 \frac{1}{r}\frac{\partial^2 u_\varphi}{\partial r \partial \varphi} - c_{32}^2 \frac{\partial \Delta_d}{\partial r}\right) \mathrm{e}^{-\alpha(t-\tau)} \mathrm{d}\tau\,, \\
\frac{\partial^2 u_\varphi}{\partial t^2} &= -2\beta\, c_{22}^2 \int_0^t \left(\frac{1}{r}\frac{\partial^2 u_r}{\partial r \partial \varphi} + \frac{\partial \omega_z}{\partial r}\right) \mathrm{e}^{-\beta\,(t-\tau)} \mathrm{d}\tau + \left(c_{31}^2 + c_{32}^2\right) \frac{1}{r}\frac{\partial \Delta_d}{\partial \varphi} + \\
&\quad +2\left(c_{21}^2 + c_{22}^2\right) \frac{\partial \omega_z}{\partial r} + \alpha \frac{1}{r}\int_0^t \left(2c_{22}^2 \frac{\partial^2 u_r}{\partial r \partial \varphi} - c_{32}^2 \frac{\partial \Delta_d}{\partial \varphi}\right) \mathrm{e}^{-\alpha(t-\tau)} \mathrm{d}\tau\,,
\end{aligned} \tag{7}$$

where $\Delta_d$ and $\omega_z$ denote volume dilatation for the state of plane strain and the rotation component corresponding to the element rotation around the axis perpendicular to the disc plane, respectively. These quantities are defined as

$$\Delta_d = \frac{\partial u_r}{\partial r} + \frac{1}{r}\left(u_r + \frac{\partial u_\varphi}{\partial \varphi}\right), \quad \omega_z = \frac{1}{2}\left[\frac{\partial u_\varphi}{\partial r} + \frac{1}{r}\left(u_\varphi - \frac{\partial u_r}{\partial \varphi}\right)\right]. \tag{8}$$

Equations (7) represent the system of two partial-integro-differential equations of the second order for the unknown functions $u_r(r, \varphi, t)$ and $u_\varphi(r, \varphi, t)$. If we compare this system with appropriate equations for the same problem of an elastic disc [5], i.e.

$$\frac{\partial^2 u_r}{\partial t^2} = {c_3}^2 \frac{\partial \Delta_d}{\partial r} - \frac{2{c_2}^2}{r}\frac{\partial \omega_z}{\partial \varphi}, \quad \frac{\partial^2 u_\varphi}{\partial t^2} = \frac{{c_3}^2}{r}\frac{\partial \Delta_d}{\partial \varphi} + 2{c_2}^2 \frac{\partial \omega_z}{\partial r}, \tag{9}$$

terms, which are the consequence of dissipative disc properties, can be easily identified. Neglecting these terms, the analogy of (7) and (9) is clear.

The formulation of appropriate initial and boundary conditions is the final step of the mathematical model derivation. With respect to the problem assumptions specified in the Section 2, the initial conditions for displacement components and their derivatives are

$$u_r|_{t=0} = 0\,, \quad u_\varphi|_{t=0} = 0\,, \quad \left.\frac{\partial u_r}{\partial t}\right|_{t=0} = 0 \quad \text{and} \quad \left.\frac{\partial u_\varphi}{\partial t}\right|_{t=0} = 0\,. \tag{10}$$

The boundary conditions can be expressed for stress components as

$$\tau_{r\varphi}|_{r=r_1} = 0 \quad \text{and} \quad \sigma_r|_{r=r_1} = -\frac{2\alpha_0\sigma_0}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin(n\alpha_0)\cos(n\varphi)}{n\alpha_0}\right) H(t)\,, \tag{11}$$

when the expansion of (1) to the Fourier cosine series was used.

## 4. Problem solution

There exist several analytical procedures how to solve the equation system (7) under conditions (10) and (11). Based on our previous experiences acquired by the investigation of analogous waves problems and with respect to the method used in [5], the combination of integral transform (namely the Laplace transform) with the Fourier (Bernoulli) method will be used.

### 4.1. The Laplace transform application

Firstly, the Laplace transform is applied to equations (7). Using the relation for the Laplace transform of convolution integral and introducing initial conditions (10), the transformed system (7) has the form

$$\begin{aligned} p^2\,\bar{u}_r &= \frac{2\,\beta\,{c_{22}}^2}{r\,(p+\beta)}\left(\frac{\partial\bar{\omega}_z}{\partial\varphi}-\frac{\partial^2\bar{u}_\varphi}{\partial r\partial\varphi}\right)+\frac{\alpha}{r\,(p+\alpha)}\left(2\,{c_{22}}^2\frac{\partial^2\bar{u}_\varphi}{\partial r\partial\varphi}-{c_{32}}^2\,r\,\frac{\partial\bar{\Delta}_d}{\partial r}\right)+\\ &\quad+\left({c_{31}}^2+{c_{32}}^2\right)\frac{\partial\bar{\Delta}_d}{\partial r}-\frac{2\left({c_{21}}^2+{c_{22}}^2\right)}{r}\,\frac{\partial\bar{\omega}_z}{\partial\varphi}\,,\\ p^2\,\bar{u}_\varphi &= -\frac{2\,\beta\,{c_{22}}^2}{r\,(p+\beta)}\left(r\,\frac{\partial\bar{\omega}_z}{\partial r}+\frac{\partial^2\bar{u}_r}{\partial r\partial\varphi}\right)+\frac{\alpha}{r\,(p+\alpha)}\left(2\,{c_{22}}^2\frac{\partial^2\bar{u}_r}{\partial r\partial\varphi}-{c_{32}}^2\,\frac{\partial\bar{\Delta}_d}{\partial\varphi}\right)+\\ &\quad+2\left({c_{21}}^2+{c_{22}}^2\right)\frac{\partial\bar{\omega}_z}{\partial r}+\frac{\left({c_{31}}^2+{c_{32}}^2\right)}{r}\,\frac{\partial\bar{\Delta}_d}{\partial\varphi}\,, \end{aligned} \tag{12}$$

where $p\in\mathcal{C}$ is the complex variable of the Laplace transform and the complex functions $\bar{u}_r\,(r,\varphi,p)$, $\bar{u}_\varphi\,(r,\varphi,p)$, $\bar{\omega}_z\,(r,\varphi,p)$ and $\bar{\Delta}_d\,(r,\varphi,p)$ correspond to the Laplace transforms of appropriate real functions mentioned above.

After making some operations with equations (12) and their terms rearrangement, the compact form of the Laplace transform of (7) may be written as

$$\begin{aligned} p^2\,r\,\bar{\omega}_z &= \left[\left(1-\frac{\beta}{p+\beta}\right)c_{22}^2+c_{21}^2\right]\left(\frac{\partial\bar{\omega}_z}{\partial r}+r\,\frac{\partial^2\bar{\omega}_z}{\partial r^2}+\frac{1}{r}\,\frac{\partial^2\bar{\omega}_z}{\partial\varphi^2}\right)+\\ &\quad+\frac{(\alpha-\beta)\,c_{22}^2\,p}{(p+\alpha)\,(p+\beta)}\left(\frac{\partial^3\bar{u}_r}{\partial r^2\partial\varphi}-\frac{1}{r}\,\frac{\partial^3\bar{u}_\varphi}{\partial r\partial\varphi^2}\right),\\ p^2\,r\,\bar{\Delta}_d &= \left[\left(1-\frac{\alpha}{p+\alpha}\right)c_{32}^2+c_{31}^2\right]\left(\frac{\partial\bar{\Delta}_d}{\partial r}+r\,\frac{\partial^2\bar{\Delta}_d}{\partial r^2}+\frac{1}{r}\,\frac{\partial^2\bar{\Delta}_d}{\partial\varphi^2}\right)+\\ &\quad+\frac{2\,(\alpha-\beta)\,c_{22}^2\,p}{(p+\alpha)\,(p+\beta)}\left(\frac{\partial^3\bar{u}_\varphi}{\partial r^2\partial\varphi}+\frac{1}{r}\,\frac{\partial^3\bar{u}_r}{\partial r\partial\varphi^2}\right). \end{aligned} \tag{13}$$

Equations (13) represent partial-differential equations for both the Laplace transforms of the displacement components $u_r$, $u_\varphi$ and the Laplace transform of the functions $\omega_z$, $\Delta_d$. But when we take into account the relation between coefficients of normal $\lambda$ and shear $\eta$ viscosity (see [13]) and the relation between $E_2$ and $G_2$, i.e.

$$\frac{\lambda}{\eta}=2\,(1+\nu)\;,\quad \frac{E_2}{G_2}=2\,(1+\mu_2)\;, \tag{14}$$

and further the assumption $\nu=\mu_2$ (see Subsection 3.1) and equations (6), we simply find out that $\alpha=\beta$. This fact leads to the essential simplification of (13), the second terms on the right hand side of both equations vanish, and we obtain two independent partial-differential equations only for $\bar{\omega}_z$ or $\bar{\Delta}_d$. Finally, defining the complex functions

$$C_2(p)=\sqrt{\left(1-\frac{\alpha}{p+\alpha}\right)c_{22}^2+c_{21}^2}\quad\text{and}\quad C_3(p)=\sqrt{\left(1-\frac{\alpha}{p+\alpha}\right)c_{32}^2+c_{31}^2}\,, \tag{15}$$

equations (13) can be rewritten into the form

$$\frac{1}{r}\,\frac{\partial\bar{\Delta}_d}{\partial r}+\frac{\partial^2\bar{\Delta}_d}{\partial r^2}+\frac{1}{r^2}\,\frac{\partial^2\bar{\Delta}_d}{\partial\varphi^2}-\frac{p^2}{C_3^2}\,\bar{\Delta}_d=0\quad\text{and}\quad \frac{1}{r}\,\frac{\partial\bar{\omega}_z}{\partial r}+\frac{\partial^2\bar{\omega}_z}{\partial r^2}+\frac{1}{r^2}\,\frac{\partial^2\bar{\omega}_z}{\partial\varphi^2}-\frac{p^2}{C_2^2}\,\bar{\omega}_z=0\,. \tag{16}$$

*4.2. The Fourier method application*

Now, we use the Fourier (Bernoulli) method for solving two independent partial-differential equations (16). We will assume that corresponding solutions may be expressed in variable separation form

$$\bar{\Delta}_d\,(r,\varphi,p) = U_1(r,p)\,\Phi_1(\varphi) \quad \text{and} \quad \bar{\omega}_z\,(r,\varphi,p) = U_2(r,p)\,\Phi_2(\varphi)\,. \tag{17}$$

Substituting supposed solutions (17) in equations (16), we obtain Bessel differential equations for the complex functions $U_1(r,p)$ and $U_2(r,p)$ and ordinary differential equations for the real functions $\Phi_1(\varphi)$ and $\Phi_2(\varphi)$. The general solutions of mentioned equations can be written as

$$\begin{aligned} U_1(r,p) &= A_1(p)\,J_n\!\left(\frac{ipr}{C_3}\right) + A_2(p)\,Y_n\!\left(\frac{ipr}{C_3}\right), \\ U_2(r,p) &= A_3(p)\,J_n\!\left(\frac{ipr}{C_2}\right) + A_4(p)\,Y_n\!\left(\frac{ipr}{C_2}\right), \\ \Phi_1(\varphi) &= B_1\cos(n\varphi) + B_2\sin(n\varphi)\,, \\ \Phi_2(\varphi) &= B_3\cos(n\varphi) + B_4\sin(n\varphi)\,, \end{aligned} \tag{18}$$

where $A_i(p)$ and $B_i$ $(i = 1,\ldots,4)$ represent integration constants and $J_n$ and $Y_n$ $(n = 0,1,2,\ldots)$ are the $n$th order Bessel functions of the first and second kind, respectively. When we take into account: (i) the properties of $J_n$ and $Y_n$ (see e.g. [16]), (ii) the displacement functions $u_r$, $u_\varphi$ reach finite values for $r = 0$, (iii) the dilatation $\Delta_d$ is an even function of $\varphi$ and (iv) the rotation $\omega_z$ is an odd function of $\varphi$, we find out that $A_2(p) = A_4(p) = 0$ and $B_2 = B_3 = 0$. Consequently, the solutions $\bar{\Delta}_d\,(r,\varphi,p)$ and $\bar{\omega}_z\,(r,\varphi,p)$ of linear equations (16) are given by

$$\bar{\Delta}_d = \sum_{n=0}^{\infty} P_n(p)\,J_n\!\left(\frac{ipr}{C_3}\right)\cos(n\varphi) \quad \text{and} \quad \bar{\omega}_z = \sum_{n=1}^{\infty} Q_n(p)\,J_n\!\left(\frac{ipr}{C_2}\right)\sin(n\varphi)\,. \tag{19}$$

Now, introducing the equality $\alpha = \beta$ together with relations (15) and (19) into modified equations (12) and after the definition of simple complex functions

$$z_1(r,p) = \frac{ipr}{C_3(p)} \quad \text{and} \quad z_2(r,p) = \frac{ipr}{C_2(p)}, \tag{20}$$

we obtain relations for required Laplace transforms:

$$\begin{aligned} \bar{u}_r(r,\varphi,p) = &-\frac{i\,C_3(p)}{p}\,J_1(z_1(r,p))\,P_0(p) - \frac{1}{r\,p^2}\sum_{n=1}^{\infty}\Big[\Big\{C_3(p)^2\Big[nJ_n(z_1(r,p)) - \\ &- z_1(r,p)J_{n-1}(z_1(r,p))\Big]P_n(p) + 2\,n\,C_2(p)^2\,J_n(z_2(r,p))\,Q_n(p)\Big\}\cos(n\varphi)\Big], \\ \bar{u}_\varphi(r,\varphi,p) = &-\frac{1}{r\,p^2}\sum_{n=1}^{\infty}\Big[\Big\{2\,C_2(p)^2\Big[nJ_n(z_2(r,p)) - z_2(r,p)J_{n-1}(z_2(r,p))\Big]Q_n(p) + \\ &+ n\,C_3(p)^2\,J_n(z_1(r,p))\,P_n(p)\Big\}\sin(n\varphi)\Big], \end{aligned} \tag{21}$$

in which the functions $P_n(p)$ $(n = 0,1,2,\ldots)$ and $Q_n(p)$ $(n = 1,2,\ldots)$ are unknown for the present.

### 4.3. The integral transforms of required functions

Equations (21) represent the Laplace transforms of the required functions $u_r$ and $u_\varphi$ in which the unknown complex functions $P_n(p)$ and $Q_n(p)$ need to be determined. The boundary conditions (11) will be used for this purpose. Owing to the fact that (11) are formulated for the stress components $\tau_{r\varphi}$ and $\sigma_r$, it is necessary to take the Laplace transform of (11) and to use (21) for the derivation of the corresponding complex functions $\bar{\tau}_{r\varphi}\,(r,\varphi,p)$ and $\bar{\sigma}_r\,(r,\varphi,p)$. Introducing (3) in (4), taking the Laplace transform of resulted relations and inserting (21) in them, one can write

$$
\begin{aligned}
\bar{\tau}_{r\varphi} &= 2\rho\, {C_2}^2 \sum_{n=1}^{\infty} \Bigg[ \Bigg\{ \left( n(n+1) \left(\frac{C_3}{pr}\right)^2 J_n(z_1(r,p)) - n\,\frac{iC_3}{pr} J_{n-1}(z_1(r,p)) \right) P_n(p) + \\
&\quad +2\left[ \left( n(n+1)\left(\frac{C_2}{pr}\right)^2 + \frac{1}{2}\right) J_n(z_2(r,p)) - \frac{iC_2}{pr} J_{n-1}(z_2(r,p)) \right] Q_n(p) \Bigg\} \sin\,(n\varphi) \Bigg], \\
\bar{\sigma}_r &= 2\rho\, {C_2}^2 \Bigg\{ \left( \frac{1}{2}\left(\frac{C_3}{C_2}\right)^2 J_0(z_1(r,p)) + \frac{iC_3}{pr} J_1(z_1(r,p)) \right) P_0(p) + \\
&\quad + \sum_{n=1}^{\infty} \Bigg[ \Bigg\{ \left[ \left( n(n+1)\left(\frac{C_3}{pr}\right)^2 + \frac{1}{2}\left(\frac{C_3}{C_2}\right)^2 \right) J_n(z_1(r,p)) - \frac{iC_3}{pr} J_{n-1}(z_1(r,p)) \right] P_n(p) + \\
&\quad +2\left( n(n+1)\left(\frac{C_2}{pr}\right)^2 J_n(z_2(r,p)) - n\,\frac{iC_2}{pr} J_{n-1}(z_2(r,p)) \right) Q_n(p) \Bigg\} \cos\,(n\varphi) \Bigg] \Bigg\}. 
\end{aligned} \tag{22}
$$

Then introducing relations (22) in the Laplace transforms of boundary conditions (11), which can be expressed as

$$
\bar{\tau}_{r\varphi}(r,\varphi,p)\big|_{r=r_1} = 0 \;\text{ and }\; \bar{\sigma}_r(r,\varphi,p)\big|_{r=r_1} = -\frac{2\alpha_0\sigma_0}{\pi p}\left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin\,(n\alpha_0)\cos\,(n\varphi)}{n\alpha_0} \right), \tag{23}
$$

we obtain the system of two equations for each $P_n(p)$ and $Q_n(p)$. Their solution can be written in the form:

$$
P_0(p) = -\frac{\sigma_0\alpha_0}{2\,\pi p\, K_{P2}(0,p)}, \quad P_n(p) = R_n(p)\, K_{Q1}(n,p) \quad \text{and} \quad Q_n(p) = -R_n(p)\, K_{P1}(n,p), \tag{24}
$$

where

$$
\begin{aligned}
R_n(p) &= \frac{K(n,p)}{K_{Q1}(n,p)\, K_{P2}(n,p) - K_{Q2}(n,p) K_{P1}(n,p)}, \\
K_{P1}(n,p) &= \frac{n}{z_1(r_1,p)} \left( J_{n-1}(z_1(r_1,p)) - \frac{(n+1)}{z_1(r_1,p)}\, J_n(z_1(r_1,p)) \right), \\
K_{P2}(n,p) &= \rho\, C_2(p)^2 \left[ \left( \frac{1}{2} K_{32}(p) - \frac{n(n+1)}{z_1(r_1,p)^2} \right) J_n(z_1(r_1,p)) + \frac{1}{z_1(r_1,p)} J_{n-1}(z_1(r_1,p)) \right], \\
K_{Q1}(n,p) &= 2\left[ \left( \frac{1}{2} - \frac{n(n+1)}{z_2(r_1,p)^2} \right) J_n(z_2(r_1,p)) + \frac{1}{z_2(r_1,p)}\, J_{n-1}(z_2(r_1,p)) \right], \\
K_{Q2}(n,p) &= 2\,\rho\, C_2(p)^2\, \frac{n}{z_2(r_1,p)} \left( J_{n-1}(z_2(r_1,p)) - \frac{(n+1)}{z_2(r_1,p)}\, J_n(z_2(r_1,p)) \right), \\
K(n,p) &= = -\frac{\sigma_0\, \sin\,(n\,\alpha_0)}{n\,\pi\, p} \quad \text{and} \quad K_{32}(p) = \left( \frac{C_3(p)}{C_2(p)} \right)^2.
\end{aligned} \tag{25}
$$

Substituting (24) and (25) into relations (21), we get the final form of the Laplace transforms $\bar{u}_r(r,\varphi,p)$ and $\bar{u}_\varphi(r,\varphi,p)$ of required displacement components. If we compare these relations with appropriate relations from [5], the analogy of terms is obvious.

As the verification of derivation process correctness, we can use the limit transition from the viscoelastic case to the elastic one. There exist several possibilities how it can be done. The easiest way is to set the material parameters equal to: $E_2 = 0$, $E_1 = E$, $G_1 = G$ and $\mu_1 = \mu$. Introducing these assumptions into the resulting equations for viscoelastic case and after some rearrangements, we obtain the same relations for $\bar{u}_r$ and $\bar{u}_\varphi$ as in [5].

The inverse Laplace transform of resulting complex functions for an elastic disc was performed analytically using the residue theorem in [5]. The author shows that the Laplace transforms have the infinite number of singular points (the poles of the first or the second order), so he expressed the solutions in time domain as the infinite sums of transforms residues in poles. The number of summed terms then corresponds to the number of dispersion curves taken into account. Contrary to the elastic problem, the situation is more complicated in the viscoelastic case. Taking the analysis of singularities of relations (21), we find out that singular points involve not only poles of the first or the second order but also branch points. It makes the analytical inverse Laplace transform much more complicated, so another method for the inversion will be probably used in future work.

## 5. Conclusion

The problem of a thin viscoelastic non-rotating disk subjected to non-stationary radial load on the part of its rim was investigated analytically in this paper. The mathematical model of the problem solved was derived for the case of standard linear viscoelastic solid model. After that, the integral transforms of the disc displacement components $u_r$ and $u_\varphi$ were derived using the combination of the Laplace transform and the Fourier (Bernoulli) method. Presented relations were derived for the case of pressure radial load with constant amplitude, but one should mention that these results are valid also for all excitations the amplitude function of which can be expanded into the cosine Fourier series. With respect to the complex form of resulting functions, the analytical inversion of the Laplace transforms obtained is relatively complicated. Consequently, other possibilities of the inversion are explored to finish the analytical solution and to obtain particular analytical results. Concretely, we focus on the following methods: (i) the substitution of Bessel functions by their definition integrals, (ii) the application of the asymptotic expansions of Bessel functions, (iii) the numerical inversion of the Laplace transforms.

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