Applied and Computational Mechanics 18 (2024) XXX–YYY

Calculation of the decomposition coefficients for plane contact problem kernel in the orthonormal basis

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Received 2 May 2023; accepted 10 January 2024

Abstract

Analytical solutions of some contact problems are infinite functional series according to the system of basic functions. When constructing such solutions, it becomes necessary to represent the kernels of integral equations describing the process of interaction in the form of two-dimensional series on a given basis. Often the kernels have a rather complex appearance, therefore, the process of finding the decomposition coefficients is a rather complex and labor-intensive process, on which the accuracy and speed of obtaining final results depend. The paper proposes a calculation method that allows calculating the coefficients of decomposition of the kernels of plane contact problems according to a special orthonormal basis that takes into account the features of contacting bodies. Other approximate formulas are also derived for the special case when coating characteristics are constant. Based on the received presentation, conclusions and recommendations are formulated.

Keywords: orthonormal basis, improper integrals, numerical-analytical methods, special functions, contact

1. Introduction

Mathematical models of many problems arising in the mechanics of contact interaction are integral equations of various types. For example, the Fredholm integral equation of the first kind describes the indention of a punch into an elastic layer as well as into an elastic wedge [1, 3, 5], the Fredholm integral equation of the second kind models the interaction of a rigid stamp and an elastic layer with a thin coating [2]. If the layer or layers are made of viscoelastic materials, then Volterra operators appear in these equations additionally [4, 13]. As can be seen from the above works, solutions for these problems can be constructed by representing the desired function as a series according to some system of orthogonal functions. In this case, the kernels of equations and other functions that are part of the equation and depend on the corresponding variables, must also be represented as series according to the system of functions included in the basis. In most cases, the kernels have a rather complex structure or have such features that do not allow the use of standard numerical methods to calculate the decomposition coefficients of these kernels on a given basis.

The paper presents approaches that allow one to calculate decomposition coefficients of kernels on a special system of eigenfunctions for plane problems of contact between a rigid punch and a coated foundation in cases when the coating has complex properties and/or the shapes of the contacting surfaces are described by rapidly changing functions. Such contact problems were considered, for example, in the works [9, 14]. Some numerical calculations and graphs are presented in these articles. Numerical calculations for plane multiple contact

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https://doi.org/10.24132/acm.2024.834

problems are presented in [8], where the calculation of the decomposition coefficients of the kernel is similar. Axisymmetric contact problems have also been considered [7]; however, the core of the Fredholm operator differs from the cores of plane problems in these cases, so the process of core decomposition will be different.

2. Integral equation and orthonormal basis

In a fairly general case, integral equations arising in plane contact and wear problems for coated base [9, 14, 15] are reduced to the following dimensionless form:

$$c(t)(\mathbf{I}-\mathbf{V}_1)m(x)q(x,t)+(\mathbf{I}-\mathbf{V}_2)\mathbf{F}q(x,t)=\delta(t)+\alpha(t)x-g(x),\ x\in[-1,1],\ t\geq 1,\quad (1)$$

where the functions $q(x,t)$, $\delta(t)$, and $\alpha(t)$ are proportional to the contact pressure under the punch, the punch settlement and its tilt angle, respectively, $m(x)$ is related to the coating rigidity and its thickness, $g(x)$ is proportional to the gap between the punch and the coating surface in undeformed state. The presence of the function $c(t)$ and the Volterra operators $\mathbf{V}_1$ and $\mathbf{V}_2$ is either due to the layers rheological properties or to the coating wear process. The operator $\mathbf{F}$ included in the equation is a positive definite Fredholm operator operating in $[-1,1]$ with a symmetric positive definite kernel

$$\mathbf{F}q(x,t)=\int\limits_{-1}^{1}k(x,\xi)\,q(\xi,t)\,\mathrm{d}\xi,\qquad k(x,\xi)=\frac{1}{\pi}\int\limits_{0}^{\infty}\frac{L(u)}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u,\qquad (2)$$

in which the function $L(u)$ depends on the type of coupling of the lower layer and the non-deformable foundation: in the case of ideal contact

$$L(u)=\frac{2\kappa\sinh(2u)-4u}{2\kappa\cosh(2u)+\kappa^2+1+4u^2},\qquad \kappa=3-4\nu,\qquad (3)$$

where ν is the Poisson's ratio, and in the case of smooth contact

$$L(u)=\frac{\cosh(2u)-1}{\sinh(2u)+2u}.\qquad (4)$$

The following properties of the function $m(x)$ should be noted, as well. This function is related to the coating rigidity and its profile and is positive over the entire interval. Moreover, for the correctness of the formulation of the mechanical problem, it is assumed that the deviations of the shape of the layer from the constant are significantly less than the thickness of the layer, and the stiffness varies within the same order. This also imposes restrictions on the function $m(x)$, which allows the use of numerical approaches described below.

In the works [9, 14, 15], it is shown that if the functions $m(x)$ and/or $g(x)$ are rapidly changing, then the solution must be constructed in a special way so that these functions are present in it as multipliers and terms. Otherwise, the solution obtained by standard methods will not allow to perform real calculations, since the number of terms of infinite series necessary to be calculated will be large. It will lead to large computational errors due to mantissa limitations.

To achieve this goal, the following steps are necessary: Firstly, using the new variables

$$Q(x,t)=\sqrt{m(x)}\,q(x,t)+(\mathbf{I}-\mathbf{V}_1)^{-1}\frac{g(x)}{c(t)\sqrt{m(x)}},\qquad K(x,\xi)=\frac{k(x,\xi)}{\sqrt{m(x)m(\xi)}},\qquad (5)$$

equation (1) should be reduced to the form

$$c(t)(\mathbf{I}-\mathbf{V}_1)Q(x,t)+(\mathbf{I}-\mathbf{V}_2)\mathbf{G}Q(x,t)=\frac{\delta(t)+\alpha(t)x-\tilde{c}(t)\tilde{g}(x)}{\sqrt{m(x)}},\ x\in[-1,1],\ t\geq 1, \quad (6)$$

where

$$\mathbf{G}Q(x,t)=\int\limits_{-1}^{1}K(x,\xi)Q(\xi,t)\,\mathrm{d}\xi, \qquad \tilde{g}(x)=\int\limits_{-1}^{1}\frac{k(x,\xi)}{m(\xi)}g(\xi)\,\mathrm{d}\xi,$$
$$\tilde{c}(t)=(\mathbf{I}-\mathbf{V}_2)(\mathbf{I}-\mathbf{V}_1)^{-1}c^{-1}(t).$$

Note that the operator $\mathbf{F}$, as well as $\mathbf{G}$, is a Fredholm operator with a positive definite symmetric kernel.

Secondly, the solution of the integral equation (6) should be constructed in the Hilbert space $L_2[-1,1]$ by a special system of basis functions $\{f_m(x)\}_{m=0,1,2,\ldots}$ obtained by orthonormalization in $L_2[-1,1]$ of the following linearly independent system:

$$\left\{\frac{1}{\sqrt{m(x)}},\frac{x}{\sqrt{m(x)}},\frac{x^2}{\sqrt{m(x)}},\ldots\right\}, \quad (7)$$

that is, the functions $f_m(x)$ are linear combinations of functions from the system (7)

$$f_m(x)=\sum_{i=0}^{m}k_{im}\frac{x^i}{\sqrt{m(x)}}.$$

If we introduce polynomials $p_m(x)$ by the rule

$$p_m(x)=\sum_{i=0}^{m}k_{im}x^i,$$

then the basic functions $f_m(x)$ will be related to them by the relation

$$f_m(x)=\frac{p_m(x)}{\sqrt{m(x)}}. \quad (8)$$

Another algorithm for constructing the solution can be found in the previously mentioned papers and in [12]. In the present paper, we will focus on the procedure for constructing the decomposition coefficients of the Fredholm operator $\mathbf{G}$ kernel into a double series according to the system of basis functions $\{f_m(x)\}_{m=0,1,2,\ldots}$. A detailed description of the method described above and a visual comparison of it with classical methods for solving contact problems is presented in [9].

3. Construction of decomposition coefficients of the kernel of a plane problem

Let us consider the kernel of the plane contact problem $K(x,\xi)$, defined by formulas $(2)_2$ and $(5)_2$. The problem is to find its decomposition coefficients into a double series by the basis functions $f_m(x)$. This can be done by double scalar multiplying the specified kernel $K(x,\xi)$ by various basis functions $f_m(x)$, $f_n(x)$ in $L_2[-1,1]$, i.e.,

$$K_{mn} = \int_{-1}^{1}\int_{-1}^{1} K(x,\xi) f_m(x) f_n(\xi)\,\mathrm{d}x\,\mathrm{d}\xi. \tag{9}$$

Given the form of the kernel $(5)_2$ and the basis functions (8), we transform (9) to the form

$$K_{mn} = \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{p_m(x)}{m(x)}\frac{p_n(\xi)}{m(\xi)} \int_{0}^{\infty} \frac{L(u)}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}\xi. \tag{10}$$

When u tends to infinity, the function $L(u)$, given by (3) or (4), tends to 1, i.e., for any predetermined accuracy ε, there is a number M such that $|L(u)-1| < \varepsilon$ for all $u > U$. Then, dividing the right part of expression (10) into two terms, we obtain an approximate expression for calculating the coefficients

$$K_{mn} \approx K_{mn}^{(U)} + K_{mn}^{(\infty)}, \tag{11}$$

where

$$\begin{aligned} K_{mn}^{(U)} &= \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{p_m(x)}{m(x)}\frac{p_n(\xi)}{m(\xi)} \int_{0}^{U} \frac{L(u)}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}\xi,\\ K_{mn}^{(\infty)} &= \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{p_m(x)}{m(x)}\frac{p_n(\xi)}{m(\xi)} \int_{U}^{\infty} \frac{1}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}\xi. \end{aligned} \tag{12}$$

Based on the approach described in [10], which consists in representing the integral function as a product of slowly and rapidly changing functions, the internal integral in (12) for the coefficients can be approximately calculated as follows:

$$\begin{aligned} \int_{0}^{U} \frac{L(u)}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u &= \sum_{k=1}^{M}\int_{u_{k-1}}^{u_k} \frac{L(u)}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u\\ &\approx \sum_{k=1}^{M} L(\bar{u}_k)\int_{u_{k-1}}^{u_k} \frac{1}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u. \end{aligned}$$

Here $\{u_k\}_{k=0,\ldots,M}$ are the u-axis coordinates such that $0 = u_0 < u_1 < \ldots < u_{M-1} < u_M = U$, $\bar{u}_k \in [u_{k-1}, u_k]$ $(k = 1,\ldots,M)$. Then

$$K_{mn}^{(U)} \approx \frac{1}{\pi}\sum_{k=1}^{M} L(\bar{u}_k)\int_{-1}^{1}\int_{-1}^{1} \frac{p_m(x)}{m(x)}\frac{p_n(\xi)}{m(\xi)} \int_{u_{k-1}}^{u_k} \frac{1}{u}\cos\left(\frac{x-\xi}{\lambda}u\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}\xi. \tag{13}$$

3.1. The general case of the function $m(x)$

First, let us consider the case when the function $m(x)$ can be either variable or constant.

Based on the definition of the cosine integral [11], the relation (13) for the coefficients $K_{mn}^{(U)}$ can be represented as

$$K_{mn}^{(U)} \approx \frac{1}{\pi} \sum_{k=1}^{M} L(\bar{u}_k) \int_{-1}^{1} \int_{-1}^{1} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \left[\mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_k \right) - \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_{k-1} \right) \right] \mathrm{d}x\, \mathrm{d}\xi,$$
$$\mathrm{Ci}(x) = -\int_{x}^{\infty} \frac{\cos u}{u}\, \mathrm{d}u. \tag{14}$$

Using the same approach as before and the properties of the function $m(x)$, approximate formulas for the double integral in (14) can also be obtained as

$$\begin{aligned} &\int_{-1}^{1} \int_{-1}^{1} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \left[\mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_k \right) - \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_{k-1} \right) \right] \mathrm{d}x\, \mathrm{d}\xi \\ &\quad = \sum_{i,j=-N+1}^{N} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \left[\mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_k \right) - \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_{k-1} \right) \right] \mathrm{d}x\, \mathrm{d}\xi \\ &\quad \approx \sum_{i,j=-N+1}^{N} \frac{p_m(\bar{x}_i)}{m(\bar{x}_i)} \frac{p_n(\bar{x}_j)}{m(\bar{x}_j)} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \left[\mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_k \right) - \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_{k-1} \right) \right] \mathrm{d}x\, \mathrm{d}\xi. \end{aligned} \tag{15}$$

Here $\{x_i\}_{i=-N,\dots,N}$ are the x-axis coordinates such that $-1 = x_{-N} < x_{-N+1} < \ldots < x_{N-1} < x_N = 1$, and $\bar{x}_i \in [x_{i-1}, x_i]$ $(i = -N+1, \dots, N)$. It can be shown that if we introduce the function

$$F(x,\xi,u) = \frac{1}{2} \left[\frac{\lambda}{u}(x-\xi) \sin\left(\frac{x-\xi}{\lambda} u \right) - \frac{\lambda^2}{u^2} \cos\left(\frac{x-\xi}{\lambda} u \right) - (x-\xi)^2\, \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u \right) \right], \tag{16}$$

then

$$\frac{\partial^2 F(x,\xi,u)}{\partial x \partial y} = \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u \right).$$

Therefore,

$$\int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u \right) \mathrm{d}x\, \mathrm{d}\xi = F(x_{i-1}, x_{j-1}, u) - F(x_{i-1}, x_j, u) - F(x_i, x_{j-1}, u) + F(x_i, x_j, u). \tag{17}$$

Then, based on (16) and (17), the double integral on the right side of (15) can be transformed to the form

$$\int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \left[\mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_k \right) - \mathrm{Ci}\left(\frac{x-\xi}{\lambda} u_{k-1} \right) \right] \mathrm{d}x\, \mathrm{d}\xi = \tilde{F}_{i-1,j-1}^{k} - \tilde{F}_{i-1,j}^{k} - \tilde{F}_{i,j-1}^{k} + \tilde{F}_{i,j}^{k}, \tag{18}$$

where

$$\begin{aligned} \tilde{F}_{i,j}^{k} &= F_{i,j}^{k} - F_{i,j}^{k-1}, \quad i,j = -N, \dots, N,\ k = 1, 2, \dots, M, \\ F_{i,j}^{k} &= F(x_i, x_j, u_k), \quad i,j = -N, \dots, N,\ k = 0, 1, \dots, M. \end{aligned} \tag{19}$$

Using formulas (14), (15), and (18), we will obtain

$$K_{mn}^{(U)} \approx \frac{1}{\pi} \sum_{i,j=-N+1}^{N} \frac{p_m(\bar{x}_i)}{m(\bar{x}_i)} \frac{p_n(\bar{x}_j)}{m(\bar{x}_j)} \left(F_{i-1,j-1}^{(U)} - F_{i-1,j}^{(U)} - F_{i,j-1}^{(U)} + F_{i,j}^{(U)} \right), \tag{20}$$

$$F_{i,j}^{(U)} = \sum_{k=1}^{M} L(\bar{u}_k) \tilde{F}_{i,j}^{k}. \tag{21}$$

Let us now consider the summand $K_{mn}^{(\infty)}$. Taking into account the definition of the cosine integral, the expression (12) for $K_{mn}^{(\infty)}$ will take the form

$$K_{mn}^{(\infty)} = -\frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \operatorname{Ci}\left(\frac{x-\xi}{\lambda} U \right) \mathrm{d}x \, \mathrm{d}\xi. \tag{22}$$

Approximate formulas for the double integral in this expression are

$$\int_{-1}^{1} \int_{-1}^{1} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \operatorname{Ci}\left(\frac{x-\xi}{\lambda} U \right) \mathrm{d}x \, \mathrm{d}\xi = \sum_{i,j=-N+1}^{N} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \frac{p_m(x)}{m(x)} \frac{p_n(\xi)}{m(\xi)} \operatorname{Ci}\left(\frac{x-\xi}{\lambda} U \right) \mathrm{d}x \, \mathrm{d}\xi$$

$$\approx \sum_{i,j=-N+1}^{N} \frac{p_m(\bar{x}_i)}{m(\bar{x}_i)} \frac{p_n(\bar{x}_j)}{m(\bar{x}_j)} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \operatorname{Ci}\left(\frac{x-\xi}{\lambda} U \right) \mathrm{d}x \, \mathrm{d}\xi. \tag{23}$$

Based on (16), (17), (22), and (23), the expression for the approximate calculation of the coefficients $K_{mn}^{(\infty)}$ has the form

$$K_{mn}^{(\infty)} \approx -\frac{1}{\pi} \sum_{i,j=-N+1}^{N} \frac{p_m(\bar{x}_i)}{m(\bar{x}_i)} \frac{p_n(\bar{x}_j)}{m(\bar{x}_j)} \left(F_{i-1,j-1}^{(\infty)} - F_{i-1,j}^{(\infty)} - F_{i,j-1}^{(\infty)} + F_{i,j}^{(\infty)} \right), \tag{24}$$

$$F_{i,j}^{(\infty)} = F(x_i, x_j, U) = F_{i,j}^{M}. \tag{25}$$

Thus, the relations (11), (16), (19)–(21), (24), and (25) allow us to obtain the following approximate formulas for calculating the decomposition coefficients of the kernel $K(x,\xi)$ into a double series according to the system of basic functions (8):

$$\begin{gathered}
K_{mn} \approx \frac{1}{\pi} \sum_{i,j=-N+1}^{N} \frac{p_m(\bar{x}_i)}{m(\bar{x}_i)} \frac{p_n(\bar{x}_j)}{m(\bar{x}_j)} \left(F_{i-1,j-1} - F_{i-1,j} - F_{i,j-1} + F_{i,j} \right), \\
F_{i,j} = F_{i,j}^{(U)} - F_{i,j}^{M}, \; F_{i,j}^{(U)} = \sum_{k=1}^{M} L(\bar{u}_k) \left(F_{i,j}^{k} - F_{i,j}^{k-1} \right), \; F_{i,j}^{k} = F(x_i, x_j, u_k), \\
F(x,\xi,u) = \frac{1}{2} \left[\frac{\lambda}{u} (x-\xi) \sin\left(\frac{x-\xi}{\lambda} u \right) - \frac{\lambda^2}{u^2} \cos\left(\frac{x-\xi}{\lambda} u \right) - (x-\xi)^2 \operatorname{Ci}\left(\frac{x-\xi}{\lambda} u \right) \right].
\end{gathered} \tag{26}$$

Note that the coefficients $F_{i,j}$ have the property of symmetry, i.e., $F_{i,j} = F_{j,i}$. If the grid is symmetrical, then $F_{i,-j} = F_{-j,i}$, and if it is also uniform, then $F_{i,j} = F_{i+1,j+1}$. These properties can be used to speed up the calculation of the decomposition coefficients.

Let us analyze the expression (26). The coefficients $F_{i,j}$ depend only on the coefficient λ and on the function $L(u)$, which, in turn, are related to the width of the stamp, the properties of the lower base (thickness and Poisson's ratio), and the conditions between the lower layer and the nondeformable foundation. The first two multipliers in the sum of (26) depend on the elastic properties of the coating and the functions describing the profiles of the contacting surfaces. Therefore, when performing calculations for cases when the properties and thickness of the lower layer, as well as the width of the stamp, do not change, it is necessary to use the same coefficients. This allows the researcher to speed up calculations for a large number of different coatings.

3.2. *Case* $m(x) = const$

Let us consider a special case when the function $m(x)$ is constant ($m(x) \equiv \hat{m}$). It is easy to show that then the basis functions $f_m(x)$ are normalized functions $\tilde{P}_m(x)$ of the Legendre polynomials $P_m(x)$

$$f_m(x) = \tilde{P}_m(x), \qquad \tilde{P}_m(x) = \sqrt{m + \frac{1}{2}}\, P_m(x). \tag{27}$$

The decomposition coefficients can be constructed both by the formulas (26), assuming that $p_m(x)/m(x) = \tilde{P}_m(x)/\sqrt{\hat{m}}$, and by applying the approach described below.

Let us convert the relations (13) and $(12)_2$ for the coefficients $K_{mn}^{(U)}$ and $K_{mn}^{(\infty)}$, respectively, to the form

$$K_{mn}^{(U)} \approx \frac{1}{\pi\hat{m}} \sum_{k=1}^{M} L(\bar{u}_k) \int_{-1}^{1}\int_{-1}^{1} \tilde{P}_m(x)\tilde{P}_n(\xi) \int_{u_{k-1}}^{u_k} \frac{1}{u} \cos\left(\frac{x-\xi}{\lambda}u\right) \mathrm{d}u\, \mathrm{d}x\, \mathrm{d}\xi,$$

$$K_{mn}^{(\infty)} = \frac{1}{\pi\hat{m}} \int_{-1}^{1}\int_{-1}^{1} \tilde{P}_m(x)\tilde{P}_n(\xi) \int_{U}^{\infty} \frac{1}{u} \cos\left(\frac{x-\xi}{\lambda}u\right) \mathrm{d}u\, \mathrm{d}x\, \mathrm{d}\xi.$$

These equations can be transformed to the form

$$\begin{aligned} K_{mn}^{(U)} \approx & \frac{1}{\pi\hat{m}} \sum_{k=1}^{M} L(\bar{u}_k) \int_{u_{k-1}}^{u_k} \frac{1}{u} \left[\int_{-1}^{1} \tilde{P}_m(x) \cos\frac{xu}{\lambda}\, \mathrm{d}x \int_{-1}^{1} \tilde{P}_n(\xi) \cos\frac{\xi u}{\lambda}\, \mathrm{d}\xi \right. \\ & \left. + \int_{-1}^{1} \tilde{P}_m(x) \sin\frac{xu}{\lambda}\, \mathrm{d}x \int_{-1}^{1} \tilde{P}_n(\xi) \sin\frac{\xi u}{\lambda}\, \mathrm{d}\xi \right] \mathrm{d}u, \\ K_{mn}^{(\infty)} = & \frac{1}{\pi\hat{m}} \int_{U}^{\infty} \frac{1}{u} \left[\int_{-1}^{1} \tilde{P}_m(x) \cos\frac{xu}{\lambda}\, \mathrm{d}x \int_{-1}^{1} \tilde{P}_n(\xi) \cos\frac{\xi u}{\lambda}\, \mathrm{d}\xi \right. \\ & \left. + \int_{-1}^{1} \tilde{P}_m(x) \sin\frac{xu}{\lambda}\, \mathrm{d}x \int_{-1}^{1} \tilde{P}_n(\xi) \sin\frac{\xi u}{\lambda}\, \mathrm{d}\xi \right] \mathrm{d}u. \end{aligned}$$

Since the integration intervals are symmetric, using the even and odd properties of normalized functions of the Legendre polynomials and trigonometric functions, we obtain

$$
\begin{aligned}
K^{(U)}_{2m,2n} &\approx \frac{1}{\pi\hat{m}} \sum_{k=1}^{M} L(\bar{u}_k) \int_{u_{k-1}}^{u_k} \frac{1}{u} \int_{-1}^{1} \tilde{P}_{2m}(x) \cos\frac{xu}{\lambda}\,\mathrm{d}x \int_{-1}^{1} \tilde{P}_{2n}(\xi) \cos\frac{\xi u}{\lambda}\,\mathrm{d}\xi\,\mathrm{d}u,\\
K^{(U)}_{2m+1,2n+1} &\approx \frac{1}{\pi\hat{m}} \sum_{k=1}^{M} L(\bar{u}_k) \int_{u_{k-1}}^{u_k} \frac{1}{u} \int_{-1}^{1} \tilde{P}_{2m+1}(x) \sin\frac{xu}{\lambda}\,\mathrm{d}x \int_{-1}^{1} \tilde{P}_{2n+1}(\xi) \sin\frac{\xi u}{\lambda}\,\mathrm{d}\xi\,\mathrm{d}u,\\
K^{(\infty)}_{2m,2n} &= \frac{1}{\pi\hat{m}} \int_{U}^{\infty} \frac{1}{u} \int_{-1}^{1} \tilde{P}_{2m}(x) \cos\frac{xu}{\lambda}\,\mathrm{d}x \int_{-1}^{1} \tilde{P}_{2n}(\xi) \cos\frac{\xi u}{\lambda}\,\mathrm{d}\xi\,\mathrm{d}u, \qquad (28)\\
K^{(\infty)}_{2m+1,2n+1} &= \frac{1}{\pi\hat{m}} \int_{U}^{\infty} \frac{1}{u} \int_{-1}^{1} \tilde{P}_{2m+1}(x) \sin\frac{xu}{\lambda}\,\mathrm{d}x \int_{-1}^{1} \tilde{P}_{2n+1}(\xi) \sin\frac{\xi u}{\lambda}\,\mathrm{d}\xi\,\mathrm{d}u,\\
K^{(U)}_{2m,2n+1} &= K^{(U)}_{2m+1,2n} = K^{(\infty)}_{2m,2n+1} = K^{(\infty)}_{2m+1,2n} = 0.
\end{aligned}
$$

Using the following integrals [6]:

$$
\int_{0}^{1} \tilde{P}_{2m}(x)\cos(ax)\,\mathrm{d}x = (-1)^m \sqrt{\frac{\pi}{2a}}\, J_{\frac{1}{2}+2m}(a),
$$

$$
\int_{0}^{1} \tilde{P}_{2m+1}(x)\sin(ax)\,\mathrm{d}x = (-1)^m \sqrt{\frac{\pi}{2a}}\, J_{\frac{3}{2}+2m}(a),
$$

where $J_s(.)$ is a Bessel function of order s, and the relation (27) between the Legendre polynomials $P_m(x)$ and the normalized functions $\tilde{P}_m(x)$, we obtain the following expressions for the coefficients K_{mn} from (11) and (28):

$$
\begin{aligned}
K_{2m,2n} &\approx (-1)^{m+n}\frac{\lambda}{\hat{m}}\sqrt{(4m+1)(4n+1)}\\
&\left[\sum_{k=1}^{M} L(\bar{u}_k) \int_{u_{k-1}}^{u_k} \frac{1}{u^2} J_{\frac{1}{2}+2m}\left(\frac{u}{\lambda}\right) J_{\frac{1}{2}+2n}\left(\frac{u}{\lambda}\right)\mathrm{d}u + \int_{U}^{\infty} \frac{1}{u^2} J_{\frac{1}{2}+2m}\left(\frac{u}{\lambda}\right) J_{\frac{1}{2}+2n}\left(\frac{u}{\lambda}\right)\mathrm{d}u\right],\\
K_{2m+1,2n+1} &\approx (-1)^{m+n}\frac{\lambda}{\hat{m}}\sqrt{(4m+3)(4n+3)} \qquad (29)\\
&\left[\sum_{k=1}^{M} L(\bar{u}_k) \int_{u_{k-1}}^{u_k} \frac{1}{u^2} J_{\frac{3}{2}+2m}\left(\frac{u}{\lambda}\right) J_{\frac{3}{2}+2n}\left(\frac{u}{\lambda}\right)\mathrm{d}u + \int_{U}^{\infty} \frac{1}{u^2} J_{\frac{3}{2}+2m}\left(\frac{u}{\lambda}\right) J_{\frac{3}{2}+2n}\left(\frac{u}{\lambda}\right)\mathrm{d}u\right],\\
K_{2m,2n+1} &= K_{2m+1,2n} = 0.
\end{aligned}
$$

The integrals in (29) can be found analytically, but the formulas for them are rather cumbersome and are not given in this article. Note that the improper integrals included in these expressions converge and that the formulas (29) coincide with those given in [4].

Thus, in the case when the coating has a constant thickness and rigidity, it is possible to use both the formulas (26), assuming that $m(x) \equiv \hat{m}$ and $p_m(x) = \sqrt{\hat{m}}\tilde{P}_m(x)$, and the formulas (29).

4. Conclusions

Approximate formulas for calculating the kernel decomposition coefficients of plane contact and wear-contact problems for foundations with coatings are derived in the work. The methods, by which the calculation of the coefficients can be accelerated, are indicated. Expressions for the coefficients are the sum of products of multipliers, one of which depends only on the parameters of the lower base and the width of the stamp. This allows for a greater number of studies of the influence of coating properties and surface profiles on contact characteristics, since most of the time is spent on calculating the kernel decomposition coefficients.

Other approximate formulas are also derived for the special case when the rigidity and thickness of the coating are constant.

Acknowledgement

The present work was supported by the Ministry of Science and Higher Education within the framework of the Russian State Assignment (contract No. 123021700050-1).

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