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Reconstruction of the material parameters of the homogenous isotropic or orthotropic 2D continuum

T. Vampola [a],*, M. Valášek [a], Z. Šika [a]

[a] *Department of Mechanics, Biomechanics and Mechatronics, Faculty of Mechanical Engineering, Czech Technical University in Prague, Karlovo nám. 13, 121 35 Prague 2, Czech Republic*

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Abstract

The paper describes the way how to find the material parameters of the homogenous isotropic or orthotropic 2D continuum from measurements of deformation of selected nodes. The FE method is used for reconstruction of the stiffness matrix. The developed method enables to determine the angle between the material axes of the continuum and the local coordinate system of the used elements.

Keywords: stiffness matrix, orthotropic material, FEM, reconstruction of parameters

1. Introduction

In the paper it is presented the way how to detect material characteristics of the 2D homogenous continuum from the deformation measurement of the defined points. Assured material characteristics are used for reconstruction of the stiffness matrix of the solved construction. In the first approximation it is assumed the linear theory of elasticity and homogenous isotropic or orthotropic material description. The example for the method demonstration is in Fig. 1.

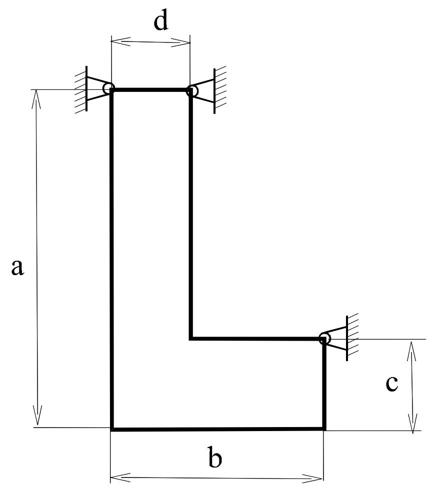

Fig. 1. Solved problem - 2D continuum.

*Corresponding author. Tel.: +422 2435 7244, e-mail: tomas.vampola@fs.cvut.cz.

2. Material description of 2D homogenous continuum

For the linear theory of elasticity the constitutive equation can be defined in the form

$$\sigma = \mathbf{C}\varepsilon . \tag{1}$$

where σ, ε are vectors of stress and strain tensors related to the original configuration of the continuum, $\mathbf{C}$ is the matrix of the elastic constants. For the isotropic material it can be assumed in case of planar stress in the form

$$\frac{\mathrm{E}}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix}, \tag{2}$$

where E is modulus of elasticity, μ is Poisson‘s constant. For the orthotropic material the matrix of the elastic constants can be found in the form [2]

$$\begin{bmatrix} \mathrm{E}_x\alpha & \mathrm{E}_y\mu_{xy}\alpha & 0 \\ \mathrm{E}_x\mu_{yx}\alpha & \mathrm{E}_y\alpha & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}, \ \alpha = \frac{1}{\mu_{xy}\mu_{yx}} . \tag{3}$$

Due to the material stability constraints for the components of the matrix of elastic constants must be fulfilled

$$\begin{aligned} &\mathrm{E}_x\alpha > 0, \mathrm{E}_y\alpha > 0, \mathrm{E}_y\mu_{xy}\alpha > 0, \mathrm{E}_x\mu_{yx}\alpha > 0, G_{xy} > 0 \\ &\left|\mathrm{E}_y\mu_{xy}\alpha\right| < \sqrt{\mathrm{E}_x / \mathrm{E}_y} \\ &\det(\mathbf{C}) > 0 \end{aligned} \tag{4}$$

Assuming validity of the expression

$$\frac{\mu_{xy}}{\mathrm{E}_x} = \frac{\mu_{yx}}{\mathrm{E}_y}, \tag{5}$$

the matrix of elastic constants for 2D orthotropic material model can be reformulated into the resulting form

$$\frac{1}{\mathrm{E}_x - \mu_{xy}^2\mathrm{E}_y}\begin{bmatrix} \mathrm{E}_x^2 & \mathrm{E}_x\,\mathrm{E}_y\mu_{xy} & 0 \\ \mathrm{E}_x\,\mathrm{E}_y\mu_{xy} & \mathrm{E}_x\mathrm{E}_y & 0 \\ 0 & 0 & \left(\mathrm{E}_x - \mu_{xy}^2\mathrm{E}_y\right)G_{xy} \end{bmatrix}. \tag{6}$$

For 2D orthotropic material model it is necessary to find the four independent material parameters: E_x, E_y are the modulus of elasticity along the material axes, μ_{xy} is Poisson‘s constant and G_{xy} is the shear modulus of elasticity.

3. Deformation characteristics of 2D homogenous continuum

For the description of the deformation characteristics of the 2D continuum FEM can be used. The linear 2D element according to Fig. 2 is accepted for the plain stress. The orientation of the material axes is assumed to be parallel to the axes of the local element coordinate system.

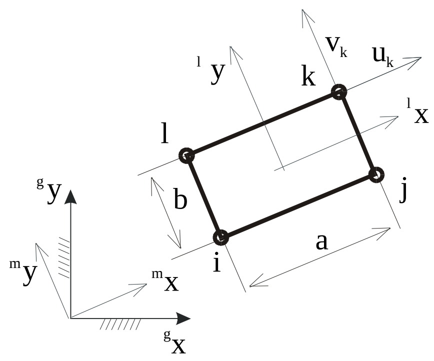

Fig. 2. 2D element.

From the condition of the deformation energy minimum the stiffness matrix of the finite element expressed by components in the local coordinate system can be derived [3]

$$U = \frac{1}{2}\int_{V_0} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, \mathrm{d}V\,. \tag{7}$$

The vector of the strain tensor $\boldsymbol{\varepsilon}$ is formulated in the form

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \tag{8}$$

where u,v are the deformations along the axes of the local coordinate system of the finite element. Due to the chosen type of the finite element the deformation field in the element is expressed by the linear interpolation of the boundary nodes deformation

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \mathbf{A}^{-1}\boldsymbol{V}\,, \tag{9}$$

where $\boldsymbol{V}$ is the vector of the deformation of the boundary nodes

$$\boldsymbol{V} = \lfloor u_i \quad v_i \quad u_j \quad v_j \quad u_k \quad v_k \quad u_l \quad v_l \rfloor\,. \tag{10}$$

The matrix **A** describes the position of the boundary nodes in the local coordinate system of the element

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5a & -0.5b & 0.25ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5a & -0.5b & 0.25ab \\ 1 & 0.5a & -0.5b & -0.25ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5a & -0.5b & -0.25ab \\ 1 & 0.5a & 0.5b & 0.25ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5a & 0.5b & 0.25ab \\ 1 & -0.5a & 0.5b & -0.25ab & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5a & 0.5b & -0.25ab \end{bmatrix}. \tag{11}$$

Substituting (9) - (12) to (8) it can be derived the expression for deformation energy of the element expressed by deformation of the boundary nodes

$$U = \frac{1}{2}\boldsymbol{V}^T \mathbf{A}^{-T} \iint \left(\Phi^T \mathbf{C} \Phi\right) \mathrm{d}x\mathrm{d}y \mathbf{A}^{-1} \boldsymbol{V}\, t\,. \tag{12}$$

where t is the thickness of the solved 2D continuum and

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix}. \tag{13}$$

The stiffness matrix of the finite element, expressed in the local coordinate system, can be formulated

$$^{loc}\mathbf{K} = \begin{bmatrix} \frac{\partial^2 U}{\partial u_i \partial u_i} & \frac{\partial^2 U}{\partial u_i \partial v_i} & \frac{\partial^2 U}{\partial u_i \partial u_j} & \frac{\partial^2 U}{\partial u_i \partial v_j} & \frac{\partial^2 U}{\partial u_i \partial u_k} & \frac{\partial^2 U}{\partial u_i \partial v_k} & \frac{\partial^2 U}{\partial u_i \partial u_l} & \frac{\partial^2 U}{\partial u_i \partial v_l} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ sym. & & & & & & & \frac{\partial^2 U}{\partial v_l \partial v_l} \end{bmatrix}. \tag{14}$$

It is apparent that the stiffness matrix is a function of the independent material parameters

$$^{loc}\mathbf{K} = {}^{loc}\mathbf{K}\left(E_x, E_y, \mu_{xy}, G_{xy}\right). \tag{15}$$

The stiffness matrix is created by 10 independent nonlinear functions. For example the expression for the function $f_1$is

$$f_1 = \frac{1}{3}\frac{E_x^2 b^2 + G_{xy} a^2 E_x - G_{xy} a^2 \mu_{xy}^2 E_y}{\left(E_x - \mu_{xy}^2\, E_y\right) ab}\,. \tag{16}$$

4. Reconstruction of material parameters of 2D homogenous continuum

For the reconstruction of the material parameters of the 2D continuum the deformations of the nodes of the patch created by finite elements according to Fig. 3 are measured.

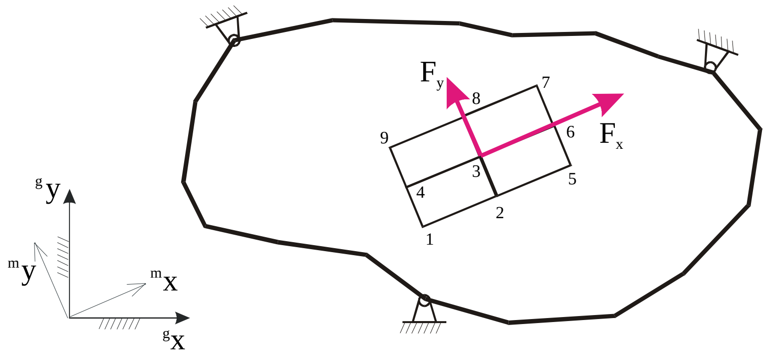

Fig. 3. Measured nodes.

The orientation of the patch is assumed to be along the direction of the material axes. Two loading cases are considered. The vectors of the deformations U_x and U_y of the patch are measured for the loading by the force F_x and F_y. The values of the deformation U_x and U_y are a function of the stiffness of the patch and stiffness and boundary conditions of the rest of the construction.

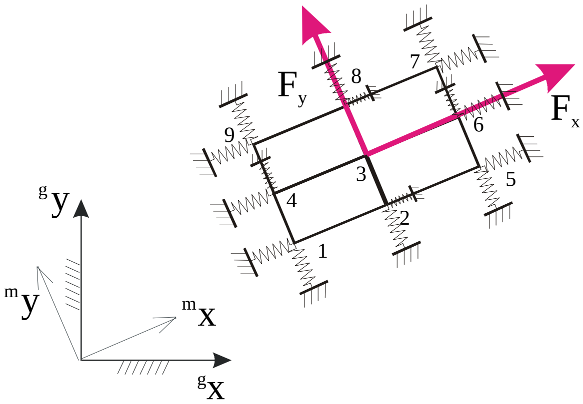

Fig. 4. Computational model.

The stiffness of the construction outside the patch is in the computational model involved by the external springs of unknown stiffness constants (Fig. 4). The external spring elements influence the stiffness matrix of the patch in the position of the main diagonal. The equilibrium equations of the computational model in Fig. 6 are

$$\begin{aligned}(\mathbf{K}(E_x, E_y, \mu_{xy}, G_{xy}) + [diag(\boldsymbol{k}_x)])\boldsymbol{U}_x &= \boldsymbol{F}_x \\ (\mathbf{K}(E_x, E_y, \mu_{xy}, G_{xy}) + [diag(\boldsymbol{k}_y)])\boldsymbol{U}_y &= \boldsymbol{F}_y\end{aligned} . \tag{17}$$

The equations (17) can be written in the symbolic form

$$\mathbf{f}(E_x, E_y, \mu_{xy}, G_{xy}, \boldsymbol{k}_x, \boldsymbol{k}_y) = \mathbf{0} . \tag{18}$$

The term (18) is composed from 36 nonlinear equations for 36 unknown parameters: E_x, E_y, μ_{xy}, G_{xy}, and 16 spring constants $\mathbf{k}_x$ and 16 spring constants $\mathbf{k}_y$ for the first and second loading cases. The modified Newton's method was used for solving (18)

$$\begin{aligned}\Delta^i \boldsymbol{x} &= -\left(\frac{\partial \mathbf{f}}{\partial \boldsymbol{x}}\right)^{-1} \mathbf{f}({}^i\boldsymbol{x}) \\ {}^{i+1}\boldsymbol{x} &\cong \boldsymbol{x}^i + \xi \Delta^i \boldsymbol{x}\end{aligned} . \tag{19}$$

where $\Delta^i\mathbf{x}$ is

$$\Delta^i \mathbf{x} = [\Delta E_x \, \Delta E_y \, \Delta\mu_{xy} \, \Delta G_{xy} \, \Delta \boldsymbol{k}_x^T \, \Delta \boldsymbol{k}_y^T]^T , \tag{20}$$

$\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is the matrix of the partial derivatives

$$\mathbf{Jac} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) = \begin{bmatrix} \dfrac{\partial \mathbf{K}}{\partial E_x}\boldsymbol{U}_x & \dfrac{\partial \mathbf{K}}{\partial E_y}\boldsymbol{U}_x & \dfrac{\partial \mathbf{K}}{\partial \mu_{xy}}\boldsymbol{U}_x & \dfrac{\partial \mathbf{K}}{\partial G_{xy}}\boldsymbol{U}_x & diag(\boldsymbol{U}_x) & \mathbf{0} \\ \dfrac{\partial \mathbf{K}}{\partial E_x}\boldsymbol{U}_y & \dfrac{\partial \mathbf{K}}{\partial E_y}\boldsymbol{U}_y & \dfrac{\partial \mathbf{K}}{\partial \mu_{xy}}\boldsymbol{U}_y & \dfrac{\partial \mathbf{K}}{\partial G_{xy}}\boldsymbol{U}_y & \mathbf{0} & diag(\boldsymbol{U}_y) \end{bmatrix} , \tag{21}$$

and $f({}^i\mathbf{x})$ is

$$\mathbf{f}\left({}^{i}\boldsymbol{x}\right)=\begin{bmatrix}\left(\mathbf{K}\left({}^{i}E_x,{}^{i}E_y,{}^{i}\mu_{xy},{}^{i}G_{xy}\right)+\left[diag\left({}^{i}\boldsymbol{k}_x\right)\right]\right)\boldsymbol{U}_x-\boldsymbol{F}_x\\ \left(\mathbf{K}\left({}^{i}E_x,{}^{i}E_y,{}^{i}\mu_{xy},{}^{i}G_{xy}\right)+\left[diag\left({}^{i}\boldsymbol{k}_y\right)\right]\right)\boldsymbol{U}_y-\boldsymbol{F}_y\end{bmatrix}. \tag{22}$$

The parameter ξ is chosen due to the validity of the expression (23)

$$\left|\mu_{xy}\right|<1. \tag{23}$$

The iteration is not too sensitive for the initial values and the speed of the convergence is apparent from Fig. 5.

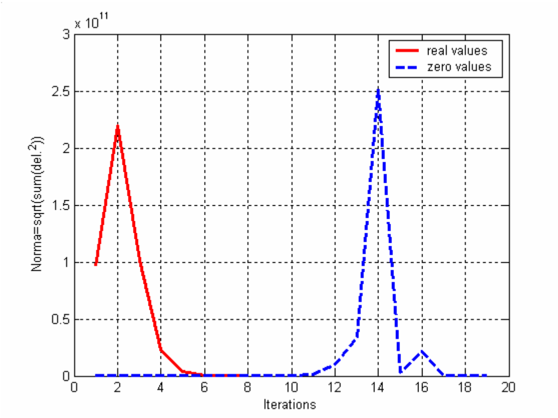

Fig. 5. Speed of iteration, real initial values/ nearly zero initial values.

The accuracy of the computed material parameters depends on the position of the measured points. Due to the numerical stability it is suitable to measure the deformation of the nodes in the position that is not too influenced by the boundary conditions.

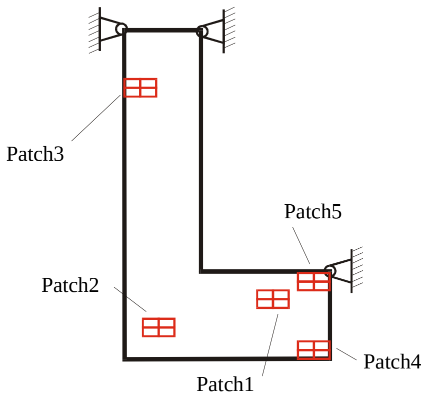

Fig. 6. Tested patches.

The results of the numerical simulations are obvious from the next table Tab. 1, where the deviations of the computed material parameter from the nominal values are presented. In the patch 5 the calculation fails due to the numerical instability and the material parameters have not been found.

Patch	ΔEx [%]	ΔEy [%]	ΔGxy [%]	Δμxy [%]
1	-0.8241	-1.0175	2.2067	0.9495
2	0.0013	-0.0032	-0.1123	0.1036
3	-0.0737	-0.0042	-0.0162	-0.0528
4	0.0111	0.0103	0.0950	-0.4888

Tab. 1. Computed material characteristics.

5. Orientation of material axes

In the case that the material axes are rotated off the local coordinate system then the stiffness matrix is the function not only of the material parameters but also of the angle of the rotation. It is necessary to transform the matrix of the elastic constants (6) to the rotated coordinate system

$$\begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} \end{bmatrix} = \begin{bmatrix} c^2\varphi & s^2\varphi & -2cs\varphi \\ s^2\varphi & c^2\varphi & 2cs\varphi \\ cs\varphi & -cs\varphi & c^2\varphi - s^2\varphi \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} \begin{bmatrix} c^2\varphi & s^2\varphi & -cs\varphi \\ s^2\varphi & c^2\varphi & cs\varphi \\ cs\varphi & -cs\varphi & c^2\varphi - s^2\varphi \end{bmatrix}^{-1} . \tag{24}$$

The expression (24) follows from the transformation of the tensors [1]. For the abbreviation of the notation $c^2\varphi=\cos^2(\varphi)$, $s^2\varphi=\sin^2(\varphi)$, $cs\varphi= \cos(\varphi)\sin(\varphi)$ was used. From the expression (24) the modified members of the matrix of the elastic constants can be derived

$$\begin{aligned} \bar{c}_{11} &= c_{11}c^4\varphi + 2(c_{12}+2c_{33})c^2\varphi\, s^2\varphi + c_{22}s^4\varphi \\ \bar{c}_{12} &= (c_{11}+c_{22}-4c_{33})c^2\varphi\, s^2\varphi + c_{12}(c^4\varphi + s^4\varphi) \\ \bar{c}_{22} &= c_{11}s^4\varphi + 2(c_{12}+2c_{33})c^2\varphi\, s^2\varphi + c_{22}c^4\varphi \\ \bar{c}_{13} &= (c_{11}-c_{22}-2c_{33})c^3\varphi\, s\varphi + (c_{12}-c_{22}+2c_{33})c\varphi\, s^3\varphi \\ \bar{c}_{23} &= (c_{11}-c_{22}-2c_{33})c\varphi\, s^3\varphi + (c_{12}-c_{22}+2c_{33})c^3\varphi\, s\varphi \\ \bar{c}_{33} &= (c_{11}+c_{22}-2c_{12}-2c_{33})c^2\varphi\, s^2\varphi + c_{33}(s^4\varphi + c^4\varphi) \end{aligned} . \tag{25}$$

According to the method presented in the paragraph 3 it can be derived the stiffness matrix

$$^{loc}\mathbf{K} = {}^{loc}\mathbf{K}(E_x, E_y, \mu_{xy}, G_{xy}, \varphi). \tag{26}$$

This matrix is created by the 20 independent nonlinear functions, where e.g. the function f_1 is

$$\begin{aligned} f1 = &\frac{1}{6ab(-E_x + \mu_{xy}^2 Ey)} \\ &\Big(2cf^2sf^2\big(a^2(-E_x^2 + 2E_xE_y\mu_{xy} - E_xE_y + 4GE_x - 2G\mu_{xy}^2E_y) + 2b^2(-2GE_x + 2G\mu_{xy}^2E_y - E_xE_y\mu_{xy})\big) \\ &+ 2cf^4(-a^2GE_x + a^2G\mu_{xy}^2E_y - b^2E_x^2) + 2sf^4(-a^2GE_x - b^2E_xE_y + a^2G\mu_{xy}^2E_y) \\ &+ 3cfsf^3(-2GE_x + 2G\mu_{xy}^2E_y - E_xE_y\mu_{xy} + E_xE_y)ba \\ &+ 3cf^3sf(E_xE_y\mu_{xy} + 2GE_x - 2G\mu_{xy}^2E_y - E_x^2)ba\Big) \end{aligned} . \tag{27}$$

The material parameters for the separate angles of the rotation of the material axes due to the local coordinate system can be computed. Excluding the solutions that do not satisfy the conditions (4), the angle of rotation of the material axes can be derived.

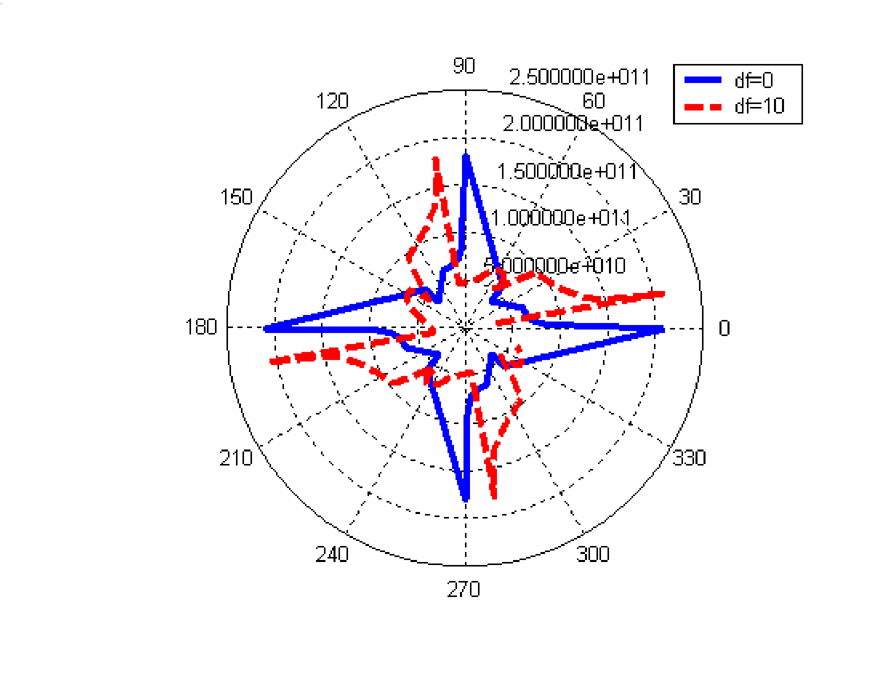

Fig. 7. Computed values of E_x, for the angle of material axes rotation df=0/df=10.

From Fig. 7 it is apparent that the values of the demanded material parameters for the correct orientation of the material axes are extremal.

6. Conclusion

In the paper it is presented the method how to find the material parameters of the isotropic or orthotropic homogenous 2D continuum from the deformation of the nodes within the assumption of the validity of the linear theory of elasticity.

Acknowledgements

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