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Diffusion and the self-measurability

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Abstract

The familiar diffusion equation, $\partial g/\partial t = D\Delta g$, is studied by using the spatially averaged quantities. A non-local relation, so-called the self-measurability condition, fulfilled by this equation is obtained. We define a broad class of diffusion equations defined by some "diffusion inequality", $\partial g/\partial t \cdot \Delta g \geq 0$, and show that it is equivalent to the self-measurability condition. It allows formulating the diffusion inequality in a non-local form. That represents an essential generalization of the diffusion problem in the case when the field $g(x,t)$ is not smooth. We derive a general differential equation for averaged quantities coming from the self-measurability condition.

Keywords: diffusion, spatial averaging, nonlocal thermomechanics

1. Introduction

The parabolic diffusion equation,

$$\frac{\partial g}{\partial t} = D\Delta g, \tag{1}$$

is very frequently used in thermomechanics. Namely, it describes a typical *dissipation* process when the time evolution of a continuum quantity g is governed by its "deviation" from a linear distribution of this quantity. Physically, if the Laplace operator equals zero at a point, $\Delta g = 0$, the local production of entropy at this point is either zero (if $g = const$) or minimal (if g is a nonconstant but linear function). A process governed by the diffusion equation tends into such a state (if it is allowed by boundary conditions) because the time evolution decreases the absolute value of the Laplace operator. The linearized heat conduction, for example, is a typical process described by (1).

The diffusion equation, however, leads to the unacceptable result that information about the distribution of quantity g propagates at infinite speed. Namely, it is a typical property of parabolic differential equations. For example, if a Dirac distribution, $g(x,t_0) = c\delta(x - x_0)$, describes the initial condition (the quantity g is zero everywhere except the point x_0) the function g becomes nonzero everywhere in an arbitrarily short time after t_0. There is a simple correction of the diffusion equation solving the problem. Namely, the addition of a second-order term, $\tau\partial^2 g/\partial t^2$, makes a hyperbolic equation from the parabolic one, whatever small is the parameter τ. This equation (proposed firstly by Cattaneo [2]),

$$\frac{\partial g}{\partial t} + \tau\frac{\partial^2 g}{\partial t^2} = D\Delta g, \tag{2}$$

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guarantees that any signal cannot propagate faster than the velocity $v_0 = \sqrt{D/\tau}$ (see e.g. [5]). The question is, however, how this correction may be explained physically. Consider the situation when the hyperbolic equation describes the heat conduction. The second-order differential equation gives an "oscillatory behavior" that means that heat may spontaneously flow (at least locally and for short time intervals) from colder to warmer regions. Though such behavior is not a violation of the second law of thermodynamics (these local processes cannot be used to make a "disallowed" effect during a thermodynamic cycle, [6]), it makes problematic any *local formulation* of thermodynamics [1, 3, 6, 7].

The problem with the infinite speed of propagation indicates that the differential equation (1) is only an *approximation* of a more proper equation. Nevertheless, we may imagine an extremely broad class of various differential equations approximating (1) in a way. This class may include linear — like the hyperbolic one with a small τ — as well as nonlinear equations. In this contribution, we define a class of differential equations by imposing only the condition that the time derivative, $\partial g/\partial t$, has the *same sign* as the the Laplacian, Δg (as trivially fulfilled at (1) because $D > 0$). That is, the class is defined by the inequality $\partial g/\partial t \cdot \Delta g \geq 0$. Though it is impossible to use effectively such an inequality, we find its equivalent formulation in a form of integral *equality*. Physically, it describes some "self-measurability" of the field $g(x,t)$. Mathematically, it is a *weak formulation* of the inequality that allows formulating the problem at non-smooth fields. Moreover, the integral formulation may be reformulated into a very general differential equation.

The paper is organized as follows. First we define the volume and surface averages of continuum quantities and recapitulate their important properties. Then the self-measurability of the standard diffusion process is found out. In next sections, the diffusion inequality is defined and it is found its equivalent integral formulation. Then the differential equation representing the integral form of the diffusion inequality is obtained.

2. Volume and surface averages

2.1. Definition

Let us have a *continuous* function f defined on d-dimensional Euclidean space E^d. Denoting as $B_l(x)$ the d-dimensional ball with the center at the point x and radius l we define two averaged values,

$$\langle f\rangle_l(x) \equiv V_l^{-1} \int_{B_l(x)} f(x')\mathrm{d}^d x', \tag{3}$$

where $V_l = K_d l^d$ is the volume of the ball ($K_1 = 2$, $K_2 = \pi$, $K_3 = 4/3\pi$ etc.), and

$$\langle f\rangle_l^b(x) \equiv S_l^{-1} \int_{\partial B_l(x)} f(x')\mathrm{d}^{d-1} x', \tag{4}$$

where $\partial B_l(x)$ is the border of the ball and $S_l = \partial V_l/\partial l = dK_d l^{d-1}$ its surface ($(d-1)$-dimensional volume). The continuity of the averaged function guarantees that $\lim_{l\to 0}\langle f\rangle_l(x) = f(x)$, whereas the volume average differs from $f(x)$ in order of l^α where $\alpha \geq 1$, i.e.

$$\langle f\rangle_l(x) = f(x) + o(l). \tag{5}$$

2.2. *Basic properties*

Useful relations connecting the averages can be obtained by introducing the origin of polar coordinate system at x and write the averages in the form

$$\langle f\rangle_l(x) \equiv V_l^{-1}\int_{\text{full sphere}}\psi(\Theta)\,\mathrm{d}\Theta\int_0^l \mathrm{d}r r^{d-1}\tilde{f}(r,\Theta), \tag{6}$$

$$\langle f\rangle_l^b(x) \equiv S_l^{-1}\int_{\text{full sphere}}\psi(\Theta)\,\mathrm{d}\Theta l^{d-1}\tilde{f}(l,\Theta) = d^{-1}K_d^{-1}\int_{\text{full sphere}}\psi(\Theta)\,\mathrm{d}\Theta\tilde{f}(l,\Theta), \tag{7}$$

where $\Theta = (\Theta_1,\ldots,\Theta_{d-1})$ are angle coordinates, $\psi(\Theta)r^{d-1}\,\mathrm{d}\Theta\,\mathrm{d}r = \mathrm{d}^d x'$, i.e.

$$V_l = K_d l^d = \int_{\text{full sphere}}\psi(\Theta)\,\mathrm{d}\Theta\int_0^l r^{d-1}\,\mathrm{d}r = d^{-1}l^d\int_{\text{full sphere}}\psi(\Theta)\,\mathrm{d}\Theta, \tag{8}$$

and $\tilde{f}(r,\Theta) = f(x')$. By using these formulas we see immediately that

$$\langle f\rangle_l = V_l^{-1}\int_0^l S_r\langle f\rangle_r^b \mathrm{d}r = dl^{-d}\int_0^l r^{d-1}\langle f\rangle_r^b\,\mathrm{d}r. \tag{9}$$

Another important equality we obtain by using the fact that

$$\frac{\partial}{\partial l}\int_0^l \mathrm{d}r r^{d-1}\tilde{f}(r,\Theta) = l^{d-1}\tilde{f}(l,\Theta). \tag{10}$$

Namely $\frac{\partial\langle f\rangle_l}{\partial l} = -\frac{\partial V_l}{\partial l}V_l^{-2}V_l\langle f\rangle_l + V_l^{-1}S_l\langle f\rangle_l^b$, and since $S_l = \partial V_l/\partial l$ we obtain the identity

$$\langle f\rangle_l^b = \langle f\rangle_l + d^{-1}l\frac{\partial\langle f\rangle_l}{\partial l}. \tag{11}$$

2.3. *Slattery-Whitaker divergence theorem*

There is a general relationship between gradients of volume averages (3) and volume averages of gradients. It was found out independently by Slattery and Whitaker [8, 11]. The relationship can be easily understood in an one-dimensional case. Namely, if $d = 1$ the volume average (3) is simply defined as

$$\langle f\rangle_l(x) = \frac{1}{2l}\int_{x-l}^{x+l} f(x')\,\mathrm{d}x'. \tag{12}$$

Let us calculate the gradient of the average, i.e. $\nabla\langle f\rangle_l = \partial\langle f\rangle_l/\partial x$. We obtain

$$\frac{\partial\langle f\rangle_l}{\partial x} = \frac{1}{2l}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{x+\varepsilon-l}^{x+\varepsilon+l} f(x')\,\mathrm{d}x' - \int_{x-l}^{x+l} f(x')\,\mathrm{d}x'\right) = \frac{1}{2l}(f(x+l) - f(x-l)). \tag{13}$$

Notice that the continuity of the function *inside* the averaged region is not necessary. Only the continuity at the border — the points $x \pm l$ — has to be demanded. Therefore we will suppose that the function f has N jump singularities at a finite set of points $x^{(j)} \in (x-l, x+l)$ and is differentiable elsewhere. The integral of the gradient is thus given by

$$\int_{x-l}^{x+l}\frac{\partial f}{\partial x}(x')\,\mathrm{d}x' = (f_-(x^{(1)}) - f(x-l)) + (f_-(x^{(2)}) - f_+(x^{(1)})) + \ldots + f(x+l), \tag{14}$$

where $f_-(x)$, $f_+(x)$ are limits of the function at x from the left and from the right respectively. Hence we have

$$\langle \nabla f \rangle_l = \frac{1}{2l}(f(x+l) - f(x-l) - \sum_{j=1}^{N} [f](x^{(j)})), \tag{15}$$

where

$$[f](x) \equiv f_+(x) - f_-(x), \tag{16}$$

and using (13) we obtain the demanded relation

$$\langle \nabla f \rangle_l = \nabla \langle f \rangle_l - \frac{1}{2l} \sum_{j=1}^{N} [f](x^{(j)}). \tag{17}$$

By using the Reynolds transport theorem a generalization of this relation in d-dimensional space can be derived,

$$\langle \nabla f \rangle_l = \nabla \langle f \rangle_l - V_l^{-1} \sum \int_{A_j \cap B} ([f] \cdot n)\,(x')\,\mathrm{d}^{d-1}x', \tag{18}$$

where $A_j \cap B$ are surfaces at which the function has a jump discontinuity within the averaged region B and n is the unit normal of surfaces pointing from f_- to f_+. It should be emphasized that ∇f on the left hand side is defined *only* outside the singular surfaces.

2.4. Correlation equality

By using the Fourier analysis a very important relation coming from volume averaging over balls have been derived recently by Voldřich [10]. The relation has an origin in the fact that shifting of averaged regions in space has to be correlated in a way with shifting the size of these regions. If the function $F(x,l) \equiv \langle f \rangle_l$ has the second derivative with respect to l, this correlation may be written in the form of differential equality, namely

$$\left(\frac{\partial^2}{\partial l^2} + (d+1) l^{-1} \frac{\partial}{\partial l} - \Delta \right) \langle f \rangle_l = 0, \tag{19}$$

where Δ is the Laplace operator with respect to x coordinates. This relation plays the important role in this work. Substituting (3) into (19) we obtain another form of this equality,

$$\left(\frac{\partial^2}{\partial l^2} - (d-1) l^{-1} \frac{\partial}{\partial l} - \Delta \right) \int_{B_l(x)} f(x')\,\mathrm{d}^d x' = 0. \tag{20}$$

Using (7) we see immediately that

$$\frac{\partial}{\partial l} \int_{B_l(x)} f(x')\,\mathrm{d} = \int \psi(\Theta)\,\mathrm{d}\Theta l^{d-1} \tilde{f}(l,\Theta) = \int_{\partial B_l(x)} f(x') \mathrm{d}^{d-1} x' \tag{21}$$

and, consequently,

$$\frac{\partial^2}{\partial l^2} \int_{B_l(x)} f(x')\,\mathrm{d} = (d-1) l^{-1} \int \psi(\Theta)\,\mathrm{d}\Theta l^{d-1} \tilde{f}(l,\Theta) + \int \psi(\Theta)\,\mathrm{d}\Theta l^{d-1} \frac{\partial \tilde{f}}{\partial l}(l,\Theta). \tag{22}$$

Using the fact that

$$\frac{\partial \tilde{f}}{\partial l} = n \cdot \nabla \tilde{f}, \tag{23}$$

where n is a unit normal vector of $\partial B_l(x)$, we obtain

$$\frac{\partial^2}{\partial l^2}\int_{B_l(x)} f(x')\,\mathrm{d} = (d-1)l^{-1}\int_{\partial B_l(x)} f(x')\,\mathrm{d}^{d-1}x' + \int_{\partial B_l(x)} n\cdot\nabla f(x')\,\mathrm{d}^{d-1}x'. \tag{24}$$

Putting (21) and (24) into (20) we get the correlation equality in the form

$$\int_{\partial B_l(x)} n\cdot\nabla f(x')\,\mathrm{d}^{d-1}x' = \Delta\left(\int_{B_l(x)} f(x')\,\mathrm{d}^d x'\right). \tag{25}$$

If the function f is *smooth enough* we can use the divergence theorem and the relation (25) gains the form

$$\int_{B_l(x)} \Delta f(x')\,\mathrm{d}^d x' = \Delta\left(\int_{B_l(x)} f(x')\,\mathrm{d}^d x'\right). \tag{26}$$

Dividing the both sides by the averaging volume we obtain the useful equality,

$$\langle \Delta f\rangle_l = \Delta\langle f\rangle_l. \tag{27}$$

Notice that the equality (25) may be understood as a generalization of the divergence theorem if the function f is not smooth within the region.

3. The standard diffusion equation and the self-measurability

We will study a process governed by the parabolic diffusion equation with a constant coefficient κ, namely

$$\kappa\frac{\partial g}{\partial t} - \Delta g = 0, \tag{28}$$

where Δ is the Laplace operator and the coefficient κ is an inverse value of the diffusion coefficient D, $\kappa \equiv D^{-1}$. The quantity g is defined on a medium in d-dimensional space and is not specified (it may be the temperature field, the density of a material component and so on).

3.1. No time evolution

If $\kappa = 0$ the equation (28) becomes

$$\Delta g = 0 \tag{29}$$

and describes the static case without a time evolution. Any solution of (29) is a *harmonic* function. These functions have an important property: the value of g at any point x equals the *averaged* value of g taken over the border of a ball with the center at x, i.e.

$$g(x) = \langle g\rangle_l^b(x). \tag{30}$$

It implies that $\langle g\rangle_l^b(x) = g(x)$ for each r and hence the equality (9) gives that

$$\langle g\rangle_l(x) = \langle g\rangle_l^b(x). \tag{31}$$

This relation has a nice interpretation: the surface integral (4) may be understood as an averaged value over a thin, ε-shell $\Sigma(\varepsilon)$ around the ball,

$$\langle g\rangle_l^b(x) \approx V_\varepsilon^{-1}\int_{\Sigma(\varepsilon)} g(x')\,\mathrm{d}^d x', \tag{32}$$

where $V_\varepsilon = \varepsilon d K_d l^{d-1}$ is the volume of $\Sigma(\varepsilon)$. The relation (31) says that information about the averaged value of g in the nearest surrounding of a small piece of media is given by the averaged value of g *inside* the piece. This is crucial when a continuum quantity is measured — measuring device gives a correct information about its surrounding.

3.2. Time evolution

Though the relation (31) concerning functions fulfilling (28) is not valid, we show that a modification of (31) can be formulated for *small* balls. To show it let us expand g into the Taylor series, i.e.

$$g(x') = g(x) + \sum_i y_i \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i,j} y_i y_j \frac{\partial^2 g}{\partial x_i \partial x_j} + \ldots, \tag{33}$$

where $y_i = x'_i - x_i$ $(i = 1, \ldots, i)$. Putting this expansion into (3) we get the volume integrals of functions such as y_i, $y_i y_j$ $(i \neq j)$ etc. These integrals are zero if the function is odd, e.g. y_i or $y_i y_j$ if $i \neq j$. Because $I_i \equiv \int y_i^2 \mathrm{d}V = I_j \equiv I$ and V_l is given by (8) we have

$$\sum_1^d I_i = dI = \int \psi(\Theta)\,\mathrm{d}\Theta \int_0^l r^{d+1}\,\mathrm{d}r = (d+2)^{-1} l^{d+2} \int \psi(\Theta)\,\mathrm{d}\Theta = V_l d l^{-d} (d+2)^{-1} l^{d+2} \tag{34}$$

and we get at the end,

$$\langle g \rangle_l = g + (1/2) l^2 (d+2)^{-1} \Delta g + o(l^4). \tag{35}$$

The surface average may be determined by using the identity (11). We have

$$\langle g \rangle_l^b = g + (1/2) l^2 d^{-1} \Delta g + o(l^4). \tag{36}$$

That is $\langle g \rangle_l$ and $\langle g \rangle_l^b$ approximates g for small l with an error $\sim l^2$. The relations (35) and (36) give

$$\langle g \rangle_l - \langle g \rangle_l^b = -l^2 d^{-1} (d+2)^{-1} \Delta g + o(l^4). \tag{37}$$

By using the equation (28) we get from (37)

$$\langle g \rangle_l - \langle g \rangle_l^b = -D_0 l^2 \frac{\partial g}{\partial t} + o(l^4), \tag{38}$$

where

$$D_0 \equiv \kappa d^{-1} (d+2)^{-1}. \tag{39}$$

The relation (5) implies that $\partial \langle g \rangle_l / \partial t = \partial g / \partial t + o(l)$. Hence when replacing $\partial g/\partial t$ by $\partial \langle g \rangle_l / \partial t$ we get

$$\langle g \rangle_l(x,t) - \langle g \rangle_l^b(x,t) = -D_0 l^2 \frac{\partial \langle g \rangle_l}{\partial t} + o(l^3). \tag{40}$$

On the other hand, by using the Taylor expansion of $\langle g \rangle_l(x,t)$ in time variable, namely

$$\langle g \rangle_l(x, t+\delta t) = \langle g \rangle_l(x,t) + \delta t \frac{\partial \langle g \rangle_l}{\partial t} + \frac{1}{2} (\delta t)^2 \frac{\partial^2 \langle g \rangle_l}{\partial t^2} + \ldots, \tag{41}$$

we can interpret $\langle g \rangle_l(x,t) + D_0 l^2 \partial \langle g \rangle_l / \partial t$ in (40) as $\langle g \rangle_l(x, t+\delta t)$ where

$$\delta t = D_0 l^2, \tag{42}$$

and obtain the equation

$$\langle g \rangle_l(x, t + \delta t(l)) = \langle g \rangle_l^b(x,t) + o(l^3), \tag{43}$$

meaning that $\langle g \rangle_l(x, t+\delta t(l)) \approx \langle g \rangle_l^b(x,t)$ for sufficiently small averaged regions (small l's).

It is a straightforward modification of (31): the averaged value of g over a spherical piece of media copies the averaged values of g over its closest surrounding (in the sense of (32)) but with a time delay δt. The diffusion process thus provides some *self-measurability* of the field $g(x,t)$: the volume averages $\langle g\rangle_l(x,t)$ give at any time instant t information about the averaged value of the field g in their nearest surroundings (represented by $\langle g\rangle_l^b$) at a previous instant $t-\delta t$. The pieces thus work as measuring device giving continuously a correct, but slightly delayed information about their surroundings. (Notice that the relation (43) may be read also in the way that the situation at the ball surface at t predicts the volume average over the ball at $t+\delta t$.)

4. The diffusion inequality

The standard diffusion equation (1) is a linear equation that expresses the spontaneous process of equilibrating the quantity g into an equilibrium state. Its validity is restricted into situations when there are no sources of the quantity g (e.g. a local heating or supply of a mass component). The process of equilibrating is demanded by the second law of thermodynamics. The *only* claim of this law is that the diffusion coefficient D cannot be negative, i.e.

$$D \geq 0. \tag{44}$$

Nevertheless, the second law does *not* restrict a possible form of the diffusion equation. We may formulate a broad class of possible differential equations (both linear and nonlinear) being in agreement with this law. There is no physical argument giving reasons for preferring the equation (1) except its extreme simplicity.

That is why we try to formulate the diffusion without using a special equation but rather as a whole class of equations defined by a condition realizing the second law. Notice that when working with the standard equation (1), the condition (44) may be written in the form

$$\frac{\partial g}{\partial t}\cdot \Delta g \geq 0. \tag{45}$$

It expresses the essential physical content of the equation (1). Namely whenever the Laplace operator is positive, $\Delta g > 0$, the time evolution *increases* the value of g. In turn, if $\Delta g < 0$ the value of g is *decreasing*.

Namely, the sign of Laplace operator measures a local distribution of the field g in the following sense: the formula (37) says that if $\Delta g > 0$ then $\langle g\rangle_l < \langle g\rangle_l^b$, if $\Delta g < 0$ then $\langle g\rangle_l > \langle g\rangle_l^b$. Hence the positivity of the Laplace operator indicates that the averaged value of the field g within a sufficiently small ball around the studied point is *smaller* than the averaged value of this field over its surface. The equilibration means that the field tends to equilibrate this "unbalance" and its value should increase. Similarly, the negativity of the Laplace operator indicates a decrease of the value of the field g. It explains the physical meaning of the relation (45) in a fully general situation (if the distribution of the quantity g may be described by a smooth function, of course).

Another reasoning for the relation (45) is as follows. Consider a body with an arbitrary distribution of the quantity g within a volume V so that the boundary conditions fix its value during time evolution, i.e.

$$g(x,t)\,|_{\text{boundary}} = g_0(x). \tag{46}$$

Obviously $\int_V (\nabla g)^2 \mathrm{d}V \geq 0$ at any time of the equilibrating process. A condition guaranteeing that the equilibrating process means a tendency of "smoothing the gradients away" may be formulated by the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\nabla g)^2 \mathrm{d}V \le 0. \tag{47}$$

Using the divergence theorem and the boundary condition (46) the condition (47) may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\nabla g)^2 \,\mathrm{d}V = 2\int_V \frac{\partial}{\partial t}(\nabla g)\nabla g \,\mathrm{d}V = -2\int_V \frac{\partial g}{\partial t}\Delta g \,\mathrm{d}V \le 0. \tag{48}$$

We see that the condition (45) guarantees the validity of (47). The question is, however, if the validity of (47) implies the validity of (45). That is why we suppose that (47) is valid for each volume V fulfilling (46) on its boundary and that there is a point x at a time t at which

$$G(x,t) \equiv \partial g/\partial t \cdot \Delta g < 0.$$

The smoothness of the field means the continuity of partial derivatives which implies that there is a region Ω including x so that $G(x',t) < 0$ for each $x' \in \Omega$. Moreover, there exists a slightly larger region Ω', $\Omega \subset \Omega'$, so that the border of Ω has a finite distance from the border of Ω' and $\int_{\Omega'} G(x',t)\,\mathrm{d}V < 0$. Imagine a physical intervention into the system at time t fixing the distribution of g at the boundary of Ω'. This intervention *cannot* influence the time evolution (values of $\partial g/\partial t$) within Ω at the same time t. Nevertheless, when fixing g on the boundary of Ω' the equalities in (48) remain valid for $V = \Omega'$ and (47) cannot be fulfilled for Ω' at time t. As a result, the validity of (45) is a necessary condition for fulfilling the inequality (47).

In what follows the condition (45) is referred to as the *diffusion inequality*. It defines a class of diffusion processes regardless if they are governed by a linear or nonlinear differential equation. It is worth stressing that the condition (45) does *not* implicate that the governing equation must have a form $F(\partial g/\partial t, \Delta g) = 0$. Namely, whatever the form of differential equation the time derivative as well as the Laplace operator may be defined at each time and point. For example, the hyperbolic diffusion equation may or may not belong into this class. In next section, we will find an integral (nonlocal) formulation of the condition (45) that allows us to formulate it in cases when the field g is not smooth. Since the nonlocal formulation is equivalent to the differential formulation when the field g is smooth, we find out, in fact, a "weak formulation" of the condition (45).

5. Nonlocal formulation

The self-measurability condition derived for the standard diffusion equation may serve as a motivation in searching for an integral formulation of the diffusion inequality (45). Let us formulate this condition as follows.

Self-measurability: At each time t and spatial point x where the field g is defined, there is a positive number l_0 and a positive real function $\delta t(l,x,t)$ fulfilling $\lim_{l\to 0}\delta t(l,x,t) = 0$, so that for each $0 < l < l_0$ there holds the condition

$$\langle g\rangle_l(x, t + \delta t(l,x,t)) = \langle g\rangle_l^b(x,t). \tag{49}$$

(Notice that the strict equality is demanded instead of (43) stating the equality up to terms of order $o(l^3)$. The reason consists in the fact that the higher order terms may be "absorbed" into the function δt (i.e. $\delta t = D_0 l^2 + o(l^3)$).) The crucial result of our study may be formulated as the following lemma.

Lemma 1. Whenever the field $g(x,t)$ is smooth in spatial and time variables the self-measurability condition is equivalent to the diffusion inequality whenever $\partial g/\partial t \cdot \Delta g \neq 0$.

Proof. First we prove that the self-measurability implies the validity of the diffusion inequality (45). Let us imagine that the condition (45) is *not* fulfilled at point x and time t, i.e. let

$$\frac{\partial g}{\partial t} \cdot \Delta g(x,t) < 0. \tag{50}$$

Let us suppose for instance that $\Delta g > 0$. The equality (37) implies that for sufficiently small l's the inequality $\langle g \rangle_l(x,t) < \langle g \rangle_l^b(x,t)$ holds. On the other hand, (50) implies that $\partial g/\partial t < 0$, i.e.

$$g(x,t+\delta t) = g(x,t) + \frac{\partial g}{\partial t}\delta t + \ldots < g(x,t) \tag{51}$$

for sufficiently small positive values δt. Since $\langle g \rangle_l \to \langle g \rangle_l^b$ if l tends to zero, the condition (51) implies that

$$\langle g \rangle_l(x,t+\delta t) < \langle g \rangle_l(x,t) \tag{52}$$

for sufficiently small balls (and sufficiently small δt). That is

$$\langle g \rangle_l(x,t+\delta t) < \langle g \rangle_l(x,t) < \langle g \rangle_l^b(x,t) \tag{53}$$

and *no* positive l exists so that (49) can be valid (since $\delta t(l) > 0$ for positive l). The case when $\Delta g < 0$ can be analyzed in a complete analogical way.

Now, let us prove that the diffusion inequality (45) implies the self-measurability. Let $\Delta g > 0$, for instance. The diffusion inequality implies that $\partial g/\partial t > 0$ (since $\partial g/\partial t \cdot \Delta g \neq 0$) that means that the function $g(x,t')$ is increasing in an interval (t,t_0). The smoothness of the field g (both in time and spatial variables) induces that there is $l' > 0$ and $t' > t$ so that for each $0 < l < l'$

$$\langle g \rangle_l(x,t+\delta t) > \langle g \rangle_l(x,t) \tag{54}$$

if $t+\delta t < t'$ (since $\lim_{l\to 0}\langle g \rangle_l(x,t) = g(x,t)$). Since $\Delta g > 0$, (37) implies that $\langle g \rangle_l(x,t) < \langle g \rangle_l^b(x,t)$ for sufficiently small l's. Moreover, the function $\langle g \rangle_l^b(x,t)$ tends to $\langle g \rangle_l(x,t)$ when $l \to 0$. It means that there is $l_0 \leq l'$ so that for any positive $l < l_0$ exists a unique $\delta t > 0$ so that the condition (49) holds. The case when $\Delta g < 0$ can be analyzed in a complete analogical way. □

The proven Lemma shows that the diffusion inequality may be formulated equivalently as the self-measurability condition if the field $g(x,t)$ is sufficiently smooth (and $\partial g/\partial t \cdot \Delta g \neq 0$). The great advantage of this formulation is its useability in situations when the field $g(x,t)$ is not smooth. Namely the averaging integrals may be defined whenever the function $g(x,t)$ is continuous. The self-measurability is thus a rather general condition defining the diffusion process.

6. The structure of differential equation yielded by the self-measurability

A surprising advantage of the self-measurability formulation of the diffusion inequality (45) is the fact that it gives a possible *structure* of any differential equation fulfilling the diffusion inequality. To show it we use the correlation equality (19). Our main result is formulated in the following Lemma.

Lemma 2. Let the function $F(x,t,l) \equiv \langle g\rangle_l(x,t)$ have the first and second spatial derivatives, all time derivatives and the first and second derivatives with respect to the length l at a point $x \in E^d$, time t and the averaging parameter l. Then the condition (49) implies the validity of the relation

$$\sum_{i=1}^{\infty} A_i \frac{\partial^i \langle g\rangle_l}{\partial t^i} = \Delta \langle g\rangle_l, \tag{55}$$

where A_i are functions uniquely determined by δt, i.e. $A_i = \hat{A}_i(\delta t(l,x,t))$, and $l < l_0$.

Proof. We use two mathematical equalities coming from the averaging over balls in d-dimensional space, namely (11) and (19). By using the expansion (41) and the identity (11) we obtain the relation (49) in the form

$$\frac{\partial \langle g\rangle_l}{\partial l} = \beta \frac{\partial \langle g\rangle_l}{\partial t} + l\beta^2 (2d)^{-1} \frac{\partial^2 \langle g\rangle_l}{\partial t^2} + \ldots, \tag{56}$$

where

$$\beta \equiv dl^{-1} \delta t(l,x,t). \tag{57}$$

Writing (56) as

$$\frac{\partial \langle g\rangle_l}{\partial l} = \sum_{i=1} b_i l^{i-1} \beta^i \frac{\partial^i \langle g\rangle_l}{\partial t^i}, \tag{58}$$

where

$$b_i = \frac{1}{i! d^{i-1}}, \tag{59}$$

and using the fact that (49) holds for each l within an interval $(0, l_0)$, we may derive (58) with respect to l, namely

$$\frac{\partial^2 \langle g\rangle_l}{\partial l^2} = \sum_{i=1} b_i l^{i-2} \beta^{i-1} \left((i-1)\beta + il \frac{\partial \beta}{\partial l} \right) \frac{\partial^i \langle g\rangle_l}{\partial t^i} + \sum_{i=1} b_i l^{i-1} \beta^i \frac{\partial^i}{\partial t^i} \left(\frac{\partial \langle g\rangle_l}{\partial l} \right). \tag{60}$$

Substituting the derivative $\partial \langle g\rangle_l / \partial l$ from (58) we obtain

$$\frac{\partial^2 \langle g\rangle_l}{\partial l^2} = \sum_{i=1} b_i l^{i-2} \beta^{i-1} \left((i-1)\beta + il \frac{\partial \beta}{\partial l} \right) \frac{\partial^i \langle g\rangle_l}{\partial t^i} + \sum_{i,j=1} b_i b_j l^{i+j-2} \beta^{i+j} \frac{\partial^{i+j} \langle g\rangle_l}{\partial t^{i+j}}. \tag{61}$$

When putting relations (58) and (61) into the equality (19) we get

$$\sum_{i=1} b_i l^{i-2} \beta^{i-1} \left((i+d)\beta + il \frac{\partial \beta}{\partial l} \right) \frac{\partial^i \langle g\rangle_l}{\partial t^i} + \sum_{i,j=1} b_i b_j l^{i+j-2} \beta^{i+j} \frac{\partial^{i+j} \langle g\rangle_l}{\partial t^{i+j}} = \Delta \langle g\rangle_l, \tag{62}$$

which is (55) where

$$\begin{aligned}
A_1 &= \partial\beta/\partial l + (d+1) l^{-1} \beta, \\
A_2 &= l d^{-1} \beta \partial\beta/\partial l + \beta^2 (3d+2)(2d)^{-1}, \\
&\ldots \\
A_i &= b_i l^{i-2} \beta^{i-1} \left((i+d)\beta + il \frac{\partial \beta}{\partial l} \right) + l^{i-2} \beta^i \sum_{m+n=i} b_m b_n \quad (i > 2,\ m,n \geq 1).
\end{aligned} \tag{63}$$

Since β is a function of δt only, the lemma is proven. □

The equation (55) is generally a highly nonlinear differential equation because the function $\delta(l, x, t)$ may depend on x, t via the values of $g(x, t)$ or its partial derivatives at x, t. Nevertheless, when neglecting this dependence we obtain a linear partial differential equation for the averaged quantity $\langle g \rangle_l(x, t)$. The continuity of the field $g(x, t)$ means that $\langle g \rangle_l(x, t)$ may be approximated by $g(x, t)$ when l tends to zero. However, the limit $l \to 0$ has to be done cautiously because we must not forget that the coefficients A_i depend on l too.

To illustrate such a limit procedure we suppose that the function δt does not depend on x, t and fulfills the condition

$$0 < \lim_{l \to 0} \frac{\delta t(l)}{l^2} < \infty, \tag{64}$$

whereas the function $g(x, t)$ defined on E_d has continuous second space derivatives and all time derivatives exist at each (x, t). It implies that the condition

$$\frac{\partial^i \langle g \rangle_l}{\partial t^i} = \left\langle \frac{\partial^i g}{\partial t^i} \right\rangle_l \tag{65}$$

is valid and (5) thus gives

$$\frac{\partial^i \langle g \rangle_l}{\partial t^i} = \frac{\partial^i g}{\partial t^i} + o(l). \tag{66}$$

Similarly, the existence of the second space derivative at each point x implies that the first derivative is a continuous function and there are no jumps in first derivatives. Due to the relations (27) and (5), it implies that

$$\Delta \langle g \rangle_l = \langle \Delta g \rangle_l = \Delta g + o(l), \tag{67}$$

because Δg is a continuous function. The equation (55) is thus equivalent to the condition

$$\sum_{i=1}^{\infty} A_i \frac{\partial^i g}{\partial t^i} = \Delta g + o(l). \tag{68}$$

The condition (64) is fulfilled by choosing

$$\delta t(l) = d^{-1}(d+1)^{-1} \kappa l^2 + o(l^3)$$

that implies that $\beta = (d+2)^{-1} \kappa l + o(l^2)$. Putting this β into the formulas (63) we get

$$A_1 = \kappa + o(l), \quad A_i = o(l^p)$$

for $i > 1$, where $p \geq 1$. Substituting these coefficients into (68) we get

$$\kappa \frac{\partial g}{\partial t} + o(l^p) = \Delta g + o(l). \tag{69}$$

Since (55) holds in an interval $(0, l_0)$ it has to be fulfilled in the limit $l \to 0$, i.e. (69) gives the parabolic diffusion equation $\kappa \partial g / \partial t = \Delta g$.

7. Conclusion

We study a broad class of diffusion equations defined by fulfilling the diffusion inequality (45). This condition describes processes which permanently decrease a "sum of squares of gradients", $\int (\nabla g)^2 \,\mathrm{dV}$. We present a new result that the diffusion inequality may be replaced by the so-called self-measurability condition that is expressed in a form of integral equality (49). This equality may be transformed into a form of a differential equation for averaged quantities (55). If the relation (64) is fulfilled, a linear limit of this equation is the standard diffusion equation. Nevertheless, other limits may be obtained when assuming another dependence of δt on l in the self-measurability conditions. As outlined in [4], one limit leads to the hyperbolic heat conduction equation. Then, however, we get outside a scope of perfectly smooth fields because the possible dependence $\delta t(l)$ includes an artificial length parameter, a, describing a regularization of the nonsmooth field $g(x,t)$, namely $\delta t(l) = A(a)l + B(a)l^2$. Hence the weak formulation is crucial. The way in which the regulation parameter is removed from the resulting equation is not trivial (a detail description appears in the work in preparation).

It is worth noting that the condition (45) cannot be generally valid. Consider for example the case when the conductivity coefficient, λ, depends on the temperature, i.e. $\lambda(T)$. Then the standard heat conduction equation obtains the form

$$\frac{\partial T}{\partial t} = c^{-1} \frac{\partial \lambda}{\partial T} (\nabla T)^2 + D \Delta T. \tag{70}$$

We see that if the Laplace operator is very small and negative, $\Delta T < 0$, the gradient of T is large and the dependence of λ on T is increasing, then the time derivative $\partial T/\partial t$ may be positive and $\Delta T \cdot \partial T/\partial t < 0$. There is an open question how broad is the group of diffusion phenomena fulfilling the self-measurability condition. Especially, its relation to the second law of thermodynamics is a very interesting question.

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